

RATIONALITY AND POINCARÉ FAMILIES FOR VECTOR BUNDLES WITH EXTRA STRUCTURE ON A CURVE

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ABSTRACT. Iterated Grassmannian bundles over moduli stacks of vector bundles on a curve are shown to be birational to an affine space times a moduli stack of degree 0 vector bundles, following the method of King and Schofield. Applications include the birational type of some Brill-Noether loci, of moduli schemes for vector bundles with parabolic structure or with level structure and for A. Schmitt's decorated vector bundles. A further consequence concerns the existence of Poincaré families on finite coverings of the moduli schemes.

INTRODUCTION

Let C be a smooth projective algebraic curve of genus $g \geq 2$, say over an algebraically closed field k , and let L be a line bundle on C . A. King and A. Schofield [21] have proved that the coarse moduli scheme $\mathfrak{Bun}_{r,L}$ of stable vector bundles E on C with $\mathrm{rk}(E) = r$ and $\det(E) \cong L$ is rational if the highest common factor h of r and $\deg(L)$ is 1. The present paper generalises this result to vector bundles with some extra structure, e. g. with parabolic structure in the sense of [30]. In that case, we obtain that the moduli scheme is rational if the highest common factor h of rank, degree and all multiplicities is 1, improving results of [5].

This highest common factor h also governs the existence of Poincaré families: Standard methods [32, Chapter 4, §5] construct e. g. a Poincaré family on the coarse moduli scheme of stable parabolic vector bundles if $h = 1$, whereas Ramanan [34] has proved that there is no Poincaré family on any open subscheme of $\mathfrak{Bun}_{r,L}$ if $h \neq 1$. We provide here a common source for such results on the rationality of coarse moduli schemes *and* on the existence of Poincaré families on them: Our main theorem 5.1 states that the moduli *stack* of fixed determinant vector bundles with extra structure is birational to an affine space \mathbb{A}^s times a moduli stack \mathcal{Bun}_{h,L_0} of rank h vector bundles with fixed determinant L_0 of degree 0.

The proof of this theorem requires to carefully keep track of something as inconspicuous as the scalar automorphisms of the vector bundles E on C . In some sense, this is already implicit in the King-Schofield proof, namely in their notion of weight [21, p. 526] and in their Brauer class $\psi_{r,d}$ – which can be interpreted as the obstruction against a Poincaré family. But the (admittedly more abstract) stack language helps to clarify and generalise here, for example to arbitrary infinite base field k (as long as C has a k -rational point). Apart from that, we basically follow the method of [21]: The moduli stack $\mathcal{Bun}_{r,L}$ is shown to be birational to a

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Grassmannian bundle over $\mathcal{B}un_{r_1, L_1}$ for some $r_1 < r$ if r does not divide $\deg(L)$; then induction is used.

A new ingredient in our proof is the refinement 3.13 of Ramanan's theorem mentioned above; it allows to extend our results to vector bundles with extra structures parameterised by iterated Grassmannian bundles over $\mathcal{B}un_{r, L}$, and it also yields that finite coverings of the coarse moduli scheme can only admit Poincaré families if their degree is a multiple of h , cf. corollary 5.3.

The structure of this text is as follows: Section 1 presents the moduli stacks that this paper deals with, and collects some basic information about them. In section 2, we introduce the notions of \mathbb{G}_m -stack and of \mathbb{G}_m -gerbe in order to systematically distinguish 'scalar automorphisms'. Section 3 and section 4 deal with vector bundles and the associated Grassmannian bundles on such stacks, respectively. Finally, section 5 contains statement and proof of the main theorem as well as some consequences and examples, including vector bundles with parabolic structure, Brill-Noether loci and A. Schmitt's decorated vector bundles.

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1. MODULI STACKS OF VECTOR BUNDLES ON A CURVE

Let k be an infinite field, and let C be a geometrically irreducible, smooth projective curve over k of genus $g \geq 2$ which has a rational point $P \in C(k)$. This paper deals with moduli spaces of vector bundles E on C from a birational point of view.

Remark 1.1. Usually, such questions have been studied over algebraically closed fields k or even over $k = \mathbb{C}$. Working over more general base fields does not make a big difference for our arguments due to lemma 1.5 below, but it has some minor technical advantages; see for example remark 1.4.ii.

By an *algebraic stack* \mathcal{M} over k , we always mean an Artin stack \mathcal{M} that is locally of finite type over k (but not necessarily quasi-compact); a standard reference for these notions is [26]. Recall that \mathcal{M} can be given by a groupoid $\mathcal{M}(S)$ for each k -scheme S , a functor $f^* : \mathcal{M}(S) \rightarrow \mathcal{M}(T)$ for each morphism of k -schemes $f : T \rightarrow S$ and isomorphisms of functors $(f \circ g)^* \cong g^* \circ f^*$. Such stacks over k form a 2-category: A 1-morphism $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ can be given by functors $\Phi(S) : \mathcal{M}(S) \rightarrow \mathcal{M}'(S)$ and isomorphisms of functors $f^* \circ \Phi(S) \cong \Phi(T) \circ f^*$; a 2-morphism $\tau : \Phi_1 \Rightarrow \Phi_2$ can be given by natural transformations $\tau(S) : \Phi_1(S) \Rightarrow \Phi_2(S)$. In all this, one can replace the base field k more generally by a noetherian ring A .

Two irreducible algebraic stacks \mathcal{M} and \mathcal{M}' over k are called *birational* if some open substacks $\emptyset \neq \mathcal{U} \subseteq \mathcal{M}$ and $\emptyset \neq \mathcal{U}' \subseteq \mathcal{M}'$ are 1-isomorphic, and *stably birational* if $\mathbb{A}^s \times \mathcal{M}$ and $\mathbb{A}^{s'} \times \mathcal{M}'$ are birational for some $s, s' \in \mathbb{N}$.

Example 1.2. We denote by $\mathcal{B}un_{r, d}$ the moduli stack of vector bundles E on C with rank $r \geq 1$ and degree $d \in \mathbb{Z}$. More precisely, $\mathcal{B}un_{r, d}(S)$ is the groupoid of all vector bundles \mathcal{E} on $C \times_k S$ with rank r and degree d over every geometric point of S , and $f^* : \mathcal{B}un_{r, d}(S) \rightarrow \mathcal{B}un_{r, d}(T)$ is 'the' usual pull-back functor.

Example 1.3. Given a line bundle L of degree d on C , we denote by $\mathcal{B}un_{r, L} \subseteq \mathcal{B}un_{r, d}$ the (reduced) closed substack of vector bundles E on C with $\det(E) \cong L$. More

precisely, $\mathcal{B}un_{r,L}(S) \subseteq \mathcal{B}un_{r,d}(S)$ is the full subgroupoid of all vector bundles \mathcal{E} on $C \times_k S$ for which $\det(\mathcal{E})$ is Zariski-locally in S isomorphic to the pull-back of L .

The basic properties of the algebraic stacks $\mathcal{B}un_{r,d}$ and $\mathcal{B}un_{r,L}$ can be found for example in [11], [16] or the appendix of [18]: They are smooth over k by standard deformation theory, $\dim_k \mathcal{B}un_{r,d} = r^2(g-1)$ and $\dim_k \mathcal{B}un_{r,L} = r^2(g-1) - g$. Moreover, they are geometrically connected, hence geometrically irreducible.

Remarks 1.4. i) Sending a vector bundle E on C to the line bundle $\det(E)$ defines a smooth morphism $\det : \mathcal{B}un_{r,d} \rightarrow \text{Pic}^d(C)$ to the Picard *scheme* of C . Its fibre over the k -point given by a line bundle L of degree d on C is precisely $\mathcal{B}un_{r,L}$.

ii) Let $K = k(\text{Pic}^d(C))$ be the function field, and let the line bundle \mathcal{L} on $C \otimes_k K$ be the generic fibre of a Poincaré family. Then the moduli stack $\mathcal{B}un_{r,\mathcal{L}}$ over K is precisely the generic fibre of \det . Thus birational results for fixed degree moduli spaces follow from their analogues for fixed determinant moduli spaces; this allows us to concentrate on the latter in the sequel.

Lemma 1.5. *Every open substack $\emptyset \neq \mathcal{U} \subseteq \mathcal{B}un_{r,L}$ contains a k -rational point.*

Proof. This follows from the well-known unirationality of $\mathcal{B}un_{r,L}$. More precisely:

We consider the open substack in $\mathcal{B}un_{r,L}$ of vector bundles E with $H^1(E) = 0$ which are generated by $H^0(E)$. Twisting by a sufficiently ample line bundle on C , we may assume that it is nonempty; we may even assume that it contains \mathcal{U} by shrinking the latter.

Now let $U \subseteq \text{Hom}(L^{\text{dual}}, \mathcal{O}_C^{r+1})$ be the open subscheme of nonzero morphisms $\varphi : L^{\text{dual}} \rightarrow \mathcal{O}_C^{r+1}$ whose cokernel is torsionfree. Sending φ to its cokernel defines a morphism $\Phi : U \rightarrow \mathcal{B}un_{r,L}$; we claim that its image contains \mathcal{U} . This implies that $\Phi^{-1}(\mathcal{U})$ is nonempty and open in the affine space $\text{Hom}(L^{\text{dual}}, \mathcal{O}_C^{r+1})$ over k , so it contains a k -rational point; its image under Φ is then a k -rational point in \mathcal{U} .

To prove the claim, we may assume that k is algebraically closed. Let E be a rank r vector bundle on C which is generated by $H^0(E)$; it suffices to construct a surjection $\mathcal{O}_C^{r+1} \twoheadrightarrow E$ since its kernel will be isomorphic to $\det(E)^{\text{dual}}$. We follow an argument of Ramanan [34, Lemma 3.1] who attributes the result to Atiyah.

Let $\Delta \subseteq C \times H^0(E)^{r+1}$ be the closed subscheme where the $r+1$ universal sections do not generate E . The restriction Δ_x of Δ to a point $x \in C(k)$ is the inverse image under the surjective evaluation map $H^0(E)^{r+1} \rightarrow E_x^{r+1}$ of all $r+1$ -tuples that do not generate the fibre $E_x \cong k^r$; by linear algebra, the latter has codimension 2 in E_x^{r+1} , so Δ_x has codimension 2 in $H^0(E)^{r+1}$. Thus Δ has codimension 2 in $C \times H^0(E)^{r+1}$, so its projection to $H^0(E)^{r+1}$ has codimension ≥ 1 . This shows that there is indeed a surjection $\mathcal{O}_C^{r+1} \twoheadrightarrow E$. \square

Remark 1.6. The above lemma implies that k -points are dense in $\mathcal{B}un_{r,d}$ if and only if they are dense in the Jacobian $J_C = \text{Pic}^0(C)$. The latter is easily seen to hold if $C(k)$ is infinite. On the other hand, an example with $J_C(k)$ finite due to Faddeev [10] is the Fermat curve $C \subseteq \mathbb{P}^2$ with equation $x^5 + y^5 = z^5$ over $k = \mathbb{Q}$.

Remark 1.7. The same arguments as for $\mathcal{B}un_{r,d}$ show in fact that the moduli stack $\mathcal{C}oh_{r,d}$ of coherent sheaves F on C with rank r and degree d is algebraic, smooth of dimension $(g-1)r^2$ over k and irreducible; cf. for example the appendix of [18]. One still has a determinant morphism $\det : \mathcal{C}oh_{r,d} \rightarrow \text{Pic}^d(C)$, e. g. by [20, p. 37].

This includes in particular the case $r = 0$ and $d \geq 0$, whereas of course $\mathcal{C}oh_{0,d} = \emptyset$ for $d < 0$. If $r \geq 1$, then $\emptyset \neq \mathcal{B}un_{r,d} \subseteq \mathcal{C}oh_{r,d}$ is an open substack, thus dense.

The following further examples of algebraic stacks will also be used in the sequel.

Example 1.8. We denote by \mathcal{Vect}_n the moduli stack of n -dimensional vector spaces over k ; more precisely, $\mathcal{Vect}_n(S)$ is the groupoid of rank n vector bundles on S . \mathcal{Vect}_n is 1-isomorphic to the stack quotient BGL_n of $\mathrm{Spec}(k)$ modulo the trivial action of GL_n ; thus it is algebraic, smooth of dimension $-n^2$ over k and irreducible.

Example 1.9. Let A be a finitely generated k -algebra, and let F_ν be for $\nu = 1, 2$ a coherent sheaf on $C \times_k \mathrm{Spec}(A)$ which is flat over A and whose restriction to each point of $\mathrm{Spec}(A)$ has rank r_ν and degree d_ν . We denote by $\mathcal{E}xt(F_2, F_1)$ the moduli stack over A of extensions of F_2 by F_1 ; more precisely, $\mathcal{E}xt(F_2, F_1)(S)$ is the following groupoid for each A -scheme $p : S \rightarrow \mathrm{Spec}(A)$:

- Its objects are the exact sequence $0 \rightarrow p^*F_1 \rightarrow \mathcal{F} \rightarrow p^*F_2 \rightarrow 0$ of coherent sheaves on $C \times_k S$; note that \mathcal{F} is automatically flat over S .
- Its morphisms are the $\mathcal{O}_{C \times_k S}$ -module isomorphisms $\mathcal{F} \rightarrow \mathcal{F}'$ that are compatible with the identity on p^*F_ν for $\nu = 1, 2$.

Lemma 1.10. *$\mathcal{E}xt(F_2, F_1)$ is algebraic, of finite type and smooth of relative dimension $(g-1)r_1r_2 + r_1d_2 - r_2d_1$ over A , with all its fibers geometrically irreducible.*

Proof. Using [20, Prop. 2.1.10], we can find an injective morphism $i : E' \hookrightarrow E$ of vector bundles on $C \times_k \mathrm{Spec}(A)$ with cokernel F_2 . One checks immediately that

$$(1) \quad \begin{array}{ccc} \mathcal{E}xt(F_2, F_1) & \longrightarrow & \mathcal{E}xt(E, F_1) \\ \downarrow & & \downarrow i^* \\ \mathrm{Spec}(A) & \xrightarrow{0} & \mathcal{E}xt(E', F_1) \end{array}$$

is a 1-cartesian diagram; here 0 is given by the trivial extension $F_1 \oplus E'$.

We choose an open affine covering $X = U \cup V$ and put $U_A := U \times_k \mathrm{Spec}(A)$, $V_A := V \times_k \mathrm{Spec}(A)$. This yields a Čech complex

$$C^\bullet(E, F_1) = \mathrm{Hom}(E|_{U_A}, F_1|_{U_A}) \oplus \mathrm{Hom}(E|_{V_A}, F_1|_{V_A}) \xrightarrow{\delta} \mathrm{Hom}(E|_{U_A \cap V_A}, F_1|_{U_A \cap V_A})$$

whose differential δ sends two morphisms α_U over U_A and α_V over V_A to $\alpha_U - \alpha_V$ over $U_A \cap V_A$. We denote the associated Picard stack [15, Exp. XVIII, 1.4] by $C^1(E, F_1)/C^0(E, F_1)$; this is by definition the stack generated by the prestack \mathcal{M} consisting of the following groupoid $\mathcal{M}(S)$ for each A -scheme $p : S \rightarrow \mathrm{Spec}(A)$:

- Its objects are the 1-cochains $\varphi \in C^1(E, F_1) \otimes_A \Gamma(S, \mathcal{O}_S)$.
- Its morphisms $\alpha : \varphi \rightarrow \psi$ are the 0-cochains $\alpha \in C^0(E, F_1) \otimes_A \Gamma(S, \mathcal{O}_S)$ with $\delta(\alpha) = \psi - \varphi$.
- Its composition law is the addition of 0-cochains.

If an object φ of $\mathcal{M}(S)$ is given, then $\mathrm{id} + \varphi$ is an automorphism of the trivial extension $p^*F_1 \oplus p^*E$ over $(U \cap V) \times_k S$; using it to glue the trivial extensions over $U \times_k S$ and over $V \times_k S$, we obtain an object of $\mathcal{E}xt(E, F_1)(S)$ which we denote by \mathcal{F}_φ . Sending φ to \mathcal{F}_φ defines a fully faithful functor

$$(2) \quad \mathcal{M}(S) \longrightarrow \mathcal{E}xt(E, F_1)(S)$$

because for every morphism $\alpha : \mathcal{F}_\varphi \rightarrow \mathcal{F}_\psi$ in $\mathcal{E}xt(E, F_1)(S)$, its restrictions α_U to $U \times_k S$ and α_V to $V \times_k S$ form a morphism $\varphi \rightarrow \psi$ in $\mathcal{M}(S)$ and conversely.

The functors (2) are compatible with pull-backs along morphisms $f : T \rightarrow S$ of schemes over A , so they induce a 1-morphism of stacks over A

$$(3) \quad C^1(E, F_1)/C^0(E, F_1) \longrightarrow \mathcal{E}xt(E, F_1)$$

which is in fact an isomorphism because (2) is essentially surjective for affine S , since p^*E is then projective over $U \times_k S$ and over $V \times_k S$. The same holds for E' instead of E , and the 1-isomorphism (3) commutes with i^* by construction.

We denote the mapping cone of $i^* : C^\bullet(E, F_1) \rightarrow C^\bullet(E', F_1)$ by $C^\bullet(F_2, F_1)$; this is a length 2 complex of flat A -modules whose cohomology is finitely generated over A by EGA III [13, 2.2.1]. Hence there is a quasi-isomorphism

$$V^\bullet = [V^0 \xrightarrow{\delta^0} V^1 \xrightarrow{\delta^1} V^2] \xrightarrow{\sim} C^\bullet(F_2, F_1)$$

from a length 2 complex of finitely generated flat A -modules V^ν , i. e. of vector bundles on $\text{Spec}(A)$; cf. [13, 6.10.5 and 7.7.12.i]. For each A -algebra B , we have $H^\nu(V^\bullet \otimes_A B) \cong \text{Ext}^\nu(F_2 \otimes_A B, F_1 \otimes_A B)$ by construction. If B is a field, then Ext^2 vanishes since C is a smooth curve; hence $\delta^1 : V^1 \rightarrow V^2$ is surjective by Nakayama's lemma, and replacing V^1 by its kernel if necessary, we may assume $V^2 = 0$. In this situation, it is straightforward to check that the diagram of Picard stacks

$$(4) \quad \begin{array}{ccc} V^1/V^0 & \longrightarrow & C^1(E, F_1)/C^0(E, F_1) \\ \downarrow & & \downarrow i^* \\ \text{Spec}(A) & \xrightarrow{0} & C^1(E', F_1)/C^0(E', F_1) \end{array}$$

is 1-cartesian as well; here V^1/V^0 is by definition the stack quotient of the total space $\text{Tot}(V^1)$ modulo the action of the algebraic group V^0/A given by δ^0 . Comparing the diagrams (1) and (4) shows that $\mathcal{E}xt(F_2, F_1)$ and V^1/V^0 are 1-isomorphic; using the Riemann-Roch formula for $\text{rk}(V^0) - \text{rk}(V^1)$, the lemma follows. \square

By a *vector bundle* \mathcal{V} of rank n on a stack \mathcal{M} over k , we mean a 1-morphism $\mathcal{V} : \mathcal{M} \rightarrow \mathcal{V}ect_n$. So \mathcal{V} assigns to each object \mathcal{E} of $\mathcal{M}(S)$ a vector bundle $\mathcal{V}(\mathcal{E})$ on the k -scheme S , in a suitably functorial way; cf. [26, Prop. 13.3.6]. More explicitly, \mathcal{V} is given by a functor $\mathcal{V}(S) : \mathcal{M}(S) \rightarrow \mathcal{V}ect_n(S)$ for each k -scheme S , and an isomorphism of functors $f^* \circ \mathcal{V}(S) \cong \mathcal{V}(T) \circ f^*$ for each k -morphism $f : T \rightarrow S$.

Examples 1.11. We fix $r_1, r_2 \geq 1$ and two line bundles L_1, L_2 on C .

i) By semicontinuity, there is an open substack $\emptyset \neq \mathcal{U} \subseteq \mathcal{B}un_{r_1, L_1} \times \mathcal{B}un_{r_2, L_2}$ of vector bundles E_1, E_2 for which $\dim \text{Ext}^1(E_1, E_2)$ is minimal, say equal to e . The vector spaces $\text{Hom}(E_1, E_2)$ are the fibers of a vector bundle $\text{Hom}(\mathcal{E}_1^{\text{univ}}, \mathcal{E}_2^{\text{univ}})$ on \mathcal{U} due to the theory of cohomology and base change in EGA III [13, Prop. 7.8.4], using that \mathcal{U} is locally noetherian and reduced. According to Riemann-Roch, its rank is $\text{rk}(\text{Hom}(\mathcal{E}_1^{\text{univ}}, \mathcal{E}_2^{\text{univ}})) = (1-g)r_1r_2 + r_1 \deg(L_2) - r_2 \deg(L_1) + e$.

ii) We moreover fix a vector bundle E_1 on C with $\text{rk}(E_1) = r_1$ and $\det(E_1) \cong L_1$. Similarly, there is an open substack $\emptyset \neq \mathcal{U}'_{E_1} \subseteq \mathcal{B}un_{r_2, L_2}$ where $\dim \text{Ext}^1(E_1, E_2)$ is minimal, say equal to e_{E_1} , and on that a vector bundle $\text{Hom}(E_1, \mathcal{E}_2^{\text{univ}})$ with fibers $\text{Hom}(E_1, E_2)$. This $\text{Hom}(E_1, \mathcal{E}_2^{\text{univ}})$ is the restriction of $\text{Hom}(\mathcal{E}_1^{\text{univ}}, \mathcal{E}_2^{\text{univ}})$ to the moduli point of E_1 if the latter is general, more precisely if $e_{E_1} = e$.

iii) Arguing as in ii for the other variable, we also obtain for each vector bundle E_2 on C a vector bundle $\text{Hom}(\mathcal{E}_1^{\text{univ}}, E_2)$ on an open substack $\emptyset \neq \mathcal{U}''_{E_2} \subseteq \mathcal{B}un_{r_1, L_1}$.

As in [21], a key ingredient to our main result will be a variant of Hirschowitz' theorem [17] that the tensor product of two general vector bundles is nonspecial. More precisely, we will use the following:

Proposition 1.12. *Given $r_1 > r_2 \geq 1$ and two line bundles L_1, L_2 on C with*

$$(5) \quad \deg(L_2)/r_2 - \deg(L_1)/r_1 > g - 1,$$

there is a surjective morphism $\varphi : E_1 \rightarrow E_2$ of vector bundles on C with $\text{rk}(E_\nu) = r_\nu$ and $\det(E_\nu) \cong L_\nu$ such that $\text{Ext}^1(E_1, E_2) = 0$.

Proof. Like [21, Lemma 2.1], this proof is similar to the one given by Russo and Teixidor [35, Thm. 1.2] for Hirschowitz' theorem itself. Since these references don't cover the fixed determinant case that we need, we recall the main arguments here.

Let $\emptyset \neq \mathcal{U} \subseteq \mathcal{Bun}_{r_1, L_1} \times \mathcal{Bun}_{r_2, L_2}$ be the open substack of vector bundles E_1 and E_2 with $\dim \text{Ext}^1(E_1, E_2)$ minimal, say equal to e . According to generic flatness [14, §6.9], there is an open substack $\emptyset \neq \mathcal{M}$ in the total space of the vector bundle $\text{Hom}(\mathcal{E}_1^{\text{univ}}, \mathcal{E}_2^{\text{univ}})$ over \mathcal{U} such that the cokernel of the universal family φ^{univ} of morphisms $\varphi : E_1 \rightarrow E_2$ is flat over \mathcal{M} . Then the image of φ^{univ} is also flat over \mathcal{M} , hence a vector bundle over $C \times_k \mathcal{M}$, say of rank r and of degree d over every point of \mathcal{M} . Since $\mathcal{M}(k) \neq \emptyset$ according to lemma 1.5, it suffices to show $e = 0$, $r = r_2$ and $d = \deg(L_2)$. Note that $r > 0$ by assumption (5) and Riemann-Roch.

By construction, \mathcal{M} is an irreducible smooth algebraic stack over k with

$$(6) \quad \dim \mathcal{M} = (g-1)(r_1^2 + r_2^2) - 2g + (1-g)r_1r_2 + r_1 \deg(L_2) - r_2 \deg(L_1) + e$$

due to Riemann-Roch. On the other hand, sending $\varphi : E_1 \rightarrow E_2$ to its kernel, image and cokernel defines a 1-morphism

$$\mathcal{M} \longrightarrow \mathcal{D} \subseteq \mathcal{Bun}_{r_1-r, \deg(L_1)-d} \times \mathcal{Bun}_{r, d} \times \mathcal{Coh}_{r_2-r, \deg(L_2)-d}$$

to the closed substack \mathcal{D} of triples F_1, F, F_2 with $\det(F_\nu) \otimes \det(F) \cong L_\nu$ for $\nu = 1, 2$. Now \mathcal{D} is irreducible of codimension $2g$ (since $r_1 - r > 0$ and $r > 0$), so

$$(7) \quad \dim \mathcal{D} = (g-1)[(r_1-r)^2 + r^2 + (r_2-r)^2] - 2g.$$

The points $\varphi : E_1 \rightarrow E_2$ in \mathcal{M} with given image in \mathcal{D} correspond to extensions

$$0 \rightarrow \ker(\varphi) \rightarrow E_1 \rightarrow \text{im}(\varphi) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{im}(\varphi) \rightarrow E_2 \rightarrow \text{coker}(\varphi) \rightarrow 0$$

that satisfy the open conditions ' E_2 torsionfree' and ' $\dim \text{Ext}^1(E_1, E_2) \leq e$ '; using lemma 1.10, this means that \mathcal{M} is smooth over \mathcal{D} of relative dimension

$$(8) \quad \begin{aligned} \dim \mathcal{M} - \dim \mathcal{D} = & (g-1)(r_1-r)r + (r_1-r)d - r(\deg L_1 - d) \\ & + (g-1)r(r_2-r) + r(\deg L_2 - d) - (r_2-r)d. \end{aligned}$$

Combining the dimension formulas (6), (7) and (8) yields

$$(9) \quad (r_1-r)(\deg L_2 - d) - (r_2-r)(\deg L_1 - d) + e = (g-1)(r_1-r)(r_2-r).$$

Now suppose $r < r_2$. The image of \mathcal{M} in $\mathcal{Bun}_{r_1, L_1} \times \mathcal{Bun}_{r_2, L_2}$ is open and dense, so it contains a pair E_1, E_2 of *stable* vector bundles on C . Consequently, the slopes $\mu_\nu := (\deg L_\nu - d)/(r_\nu - r)$ of $\ker(\varphi)$ and $\text{coker}(\varphi)$ satisfy $\mu_1 < \deg(L_1)/r_1$ and $\deg(L_2)/r_2 < \mu_2$. On the other hand, $\mu_2 - \mu_1 \leq g-1$ by equation (9). This leads to a contradiction with our assumption (5), thereby proving $r = r_2$.

Now equation (9) reads $(r_1-r)(\deg L_2 - d) + e = 0$. But $d \leq \deg(L_2)$ because otherwise $\mathcal{Coh}_{r_2-r, \deg(L_2)-d}$ would be empty, $r_1 - r \geq r_1 - r_2 > 0$ by hypothesis and of course also $e \geq 0$; thus we conclude $\deg(L_2) = d$ and $e = 0$. \square

2. \mathbb{G}_m -STACKS AND \mathbb{G}_m -GERBES

Definition 2.1. A \mathbb{G}_m -stack $\mathcal{M} = (\mathcal{M}, \iota)$ over k consists of an algebraic stack \mathcal{M} over k together with a group homomorphism $\iota(\mathcal{E}) : \Gamma(S, \mathcal{O}_S^*) \rightarrow \text{Aut}_{\mathcal{M}(S)}(\mathcal{E})$ for each k -scheme S and each object \mathcal{E} of the groupoid $\mathcal{M}(S)$ such that the diagrams

$$\begin{array}{ccc} \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\iota(\mathcal{E})} & \text{Aut}_{\mathcal{M}(S)}(\mathcal{E}) \\ & \searrow \iota(\mathcal{E}') & \downarrow \alpha \mapsto \varphi \alpha \varphi^{-1} \\ & & \text{Aut}_{\mathcal{M}(S)}(\mathcal{E}') \end{array} \quad \text{and} \quad \begin{array}{ccc} \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\iota(\mathcal{E})} & \text{Aut}_{\mathcal{M}(S)}(\mathcal{E}) \\ f^* \downarrow & & \downarrow f^* \\ \Gamma(T, \mathcal{O}_T^*) & \xrightarrow{\iota(f^* \mathcal{E})} & \text{Aut}_{\mathcal{M}(T)}(f^* \mathcal{E}) \end{array}$$

commute for each morphism $\varphi : \mathcal{E} \rightarrow \mathcal{E}'$ in $\mathcal{M}(S)$ and each k -morphism $f : T \rightarrow S$.

Examples 2.2. The moduli stacks of vector bundles $\mathcal{Bun}_{r,d}$ and $\mathcal{Bun}_{r,L}$ are in a canonical way \mathbb{G}_m -stacks: The multiplication by scalars defines a canonical homomorphism $\iota(\mathcal{E}) : \Gamma(S, \mathcal{O}_S^*) \rightarrow \text{Aut}(\mathcal{E})$ for each k -scheme S and each vector bundle \mathcal{E} on $C \times_k S$; the functoriality conditions above are obviously satisfied.

Definition 2.3. A \mathbb{G}_m -stack (\mathcal{M}, ι) is a \mathbb{G}_m -gerbe if $\iota(\mathcal{E}) : \Gamma(S, \mathcal{O}_S^*) \rightarrow \text{Aut}_{\mathcal{M}(S)}(\mathcal{E})$ is an isomorphism for every k -scheme S and every object \mathcal{E} of $\mathcal{M}(S)$.

Examples 2.4. The open substacks $\mathcal{Bun}_{r,d}^{\text{stab}} \subseteq \mathcal{Bun}_{r,d}$ and $\mathcal{Bun}_{r,L}^{\text{stab}} \subseteq \mathcal{Bun}_{r,L}$ of geometrically stable vector bundles E on C are \mathbb{G}_m -gerbes.

Proof. Geometrically stable vector bundles E in C are known to be *simple*, i.e. $\text{End}(E) = k \cdot \text{id}_E$. If S is a k -scheme and \mathcal{E} is a vector bundle on $C \times_k S$ whose restriction to each point of S is simple, then the theory of cohomology and base change in EGA III [13] shows $\text{End}(\mathcal{E}) = \Gamma(S, \mathcal{O}_S) \cdot \text{id}_{\mathcal{E}}$. (Strictly speaking, [13] applies only if S is locally noetherian, but standard limit arguments allow us to assume that.) Hence $\iota(\mathcal{E}) : \Gamma(S, \mathcal{O}_S^*) \rightarrow \text{Aut}(\mathcal{E})$ is indeed an isomorphism. \square

Example 2.5. If \mathfrak{M} is an algebraic space locally of finite type over k , then the stack $\mathcal{M} := \mathfrak{M} \times_k \text{B}\mathbb{G}_m$ is a \mathbb{G}_m -gerbe over k in an obvious way.

Definition 2.6. Let (\mathcal{M}, ι) and (\mathcal{M}', ι') be \mathbb{G}_m -stacks over k . A 1-morphism $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ has weight $w \in \mathbb{Z}$ if the diagram

$$\begin{array}{ccc} \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\iota(\mathcal{E})} & \text{Aut}_{\mathcal{M}(S)}(\mathcal{E}) \\ (-)^w \downarrow & & \downarrow \Phi(S) \\ \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\iota'(\Phi(\mathcal{E}))} & \text{Aut}_{\mathcal{M}'(S)}(\Phi(\mathcal{E})) \end{array}$$

commutes for every k -scheme S and every object \mathcal{E} of the groupoid $\mathcal{M}(S)$.

Example 2.7. There is a canonical 1-isomorphism $\mathcal{Bun}_{r,L} \rightarrow \mathcal{Bun}_{r,L}^{\text{dual}}$ of stacks over k which has weight -1 : It sends a vector bundle \mathcal{E} on $C \times_k S$ to $\mathcal{E}^{\text{dual}}$ and an isomorphism $\varphi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ of vector bundles on $C \times_k S$ to $(\varphi^{-1})^{\text{dual}} : \mathcal{E}_1^{\text{dual}} \rightarrow \mathcal{E}_2^{\text{dual}}$.

Lemma 2.8. Let (\mathcal{M}, ι) and (\mathcal{M}', ι') be \mathbb{G}_m -gerbes over k . If \mathcal{M} is connected, then every 1-isomorphism $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ has weight 1 or has weight -1 .

Proof. Let S be a k -scheme, and let \mathcal{E} be an object of $\mathcal{M}(S)$. For each morphism of k -schemes $f : T \rightarrow S$, we consider the group isomorphism

$$\Gamma(T, \mathcal{O}_T^*) \cong \text{Aut}_{\mathcal{M}(T)}(f^* \mathcal{E}) \xrightarrow{\Phi(T)} \text{Aut}_{\mathcal{M}'(T)}(\Phi(f^* \mathcal{E})) \cong \Gamma(T, \mathcal{O}_T).$$

As T varies, these form an automorphism of the group scheme \mathbb{G}_m over S . Hence S is a disjoint union of open subschemes S_+ and S_- where this automorphism is the identity and $\lambda \mapsto \lambda^{-1}$, respectively. This shows that \mathcal{M} is the disjoint union of open substacks \mathcal{M}_+ and \mathcal{M}_- where Φ has weight 1 and weight -1 , respectively; since \mathcal{M} is connected, one of these is empty. \square

Recall that the *coarse space* \mathfrak{M} associated to an algebraic stack \mathcal{M} over k is the fppf-sheafification of the functor $(\text{Schemes}/k)^{\text{op}} \rightarrow \text{Sets}$ that sends each k -scheme S to the set of isomorphism classes in the groupoid $\mathcal{M}(S)$. We say that \mathfrak{M} is a scheme or an algebraic space if this sheafified functor is representable by such.

Examples 2.9. The coarse spaces associated to the \mathbb{G}_m -gerbes $\mathcal{Bun}_{r,d}^{\text{stab}}$ and $\mathcal{Bun}_{r,L}^{\text{stab}}$ are the usual quasiprojective coarse moduli schemes $\mathfrak{Bun}_{r,d}$ and $\mathfrak{Bun}_{r,L}$ of stable vector bundles E on C . We denote by $\mathfrak{Bun}_{r,d} \subseteq \overline{\mathfrak{Bun}}_{r,d}$ and by $\mathfrak{Bun}_{r,L} \subseteq \overline{\mathfrak{Bun}}_{r,L}$ the natural compactifications given by semistable vector bundles.

Remark 2.10. Each \mathbb{G}_m -gerbe \mathcal{M} in the sense above is a gerbe with band \mathbb{G}_m over its coarse space \mathfrak{M} in the sense of [12, Déf. IV.2.2.2]; cf. also [26, Rem. 3.19]. This justifies the terminology ‘ \mathbb{G}_m -gerbe’.

Lemma 2.11. *Let \mathfrak{M} be the coarse space associated to a \mathbb{G}_m -gerbe \mathcal{M} over k .*

- i) \mathfrak{M} is an algebraic space locally of finite type over k .*
- ii) The canonical 1-morphism $\pi : \mathcal{M} \rightarrow \mathfrak{M}$ is faithfully flat of finite presentation.*
- iii) The induced map of points $|\pi| : |\mathcal{M}| \rightarrow |\mathfrak{M}|$ is a homeomorphism for the Zariski topologies defined by [25, Def. II.6.9] and [26, 5.5].*

Proof. Due to [26, Cor. 10.8], \mathfrak{M} is an algebraic space, and π is faithfully flat and locally of finite presentation. The induced map $|\pi| : |\mathcal{M}| \rightarrow |\mathfrak{M}|$ is automatically continuous, and it is bijective by the definition [26, Déf. 5.2] of point; $|\pi|$ is also open since π is flat [26, Prop. 5.6]. This proves iii; in particular, π is quasicompact, which completes the proof of ii. Using [14, 2.7.1 and 17.7.5], ii implies that \mathfrak{M} is locally of finite type over k because \mathcal{M} is so, by our convention on stacks. \square

Remarks 2.12. i) The coarse space $\pi : \mathcal{M} \rightarrow \mathfrak{M}$ associated to a \mathbb{G}_m -gerbe \mathcal{M} over k is by definition universal for morphisms from \mathcal{M} to algebraic spaces over k ; in particular, \mathfrak{M} is reduced if \mathcal{M} is. Part iii of the above lemma 2.11 implies that \mathfrak{M} is irreducible if \mathcal{M} is. Hence \mathfrak{M} is integral if \mathcal{M} is.

ii) If irreducible \mathbb{G}_m -gerbes \mathcal{M} and \mathcal{M}' over k are birational, then their coarse spaces \mathfrak{M} and \mathfrak{M}' are birational as well, due to the above lemma 2.11.iii again.

A \mathbb{G}_m -gerbe \mathcal{M} is called *neutral* if the canonical 1-morphism to its coarse space $\pi : \mathcal{M} \rightarrow \mathfrak{M}$ admits a section $s : \mathfrak{M} \rightarrow \mathcal{M}$. If this is the case, then there is a 1-isomorphism $\mathcal{M} \cong \mathfrak{M} \times \text{B}\mathbb{G}_m$ of weight 1 according to [26, Lemme 3.21].

Remark 2.13. Lemma 2.11.ii above implies that every \mathbb{G}_m -gerbe \mathcal{M} over k is locally neutral for the fppf-topology on its coarse space \mathfrak{M} .

Definition 2.14. Let $\pi : \mathcal{M} \rightarrow \mathfrak{M}$ be the coarse space associated to a \mathbb{G}_m -gerbe \mathcal{M} over k , and let $p : S \rightarrow \mathfrak{M}$ be a scheme over \mathfrak{M} . A *Poincaré family* on S is an object \mathcal{E} of $\mathcal{M}(S)$ whose classifying morphism $c_{\mathcal{E}} : S \rightarrow \mathcal{M}$ satisfies $p = \pi \circ c_{\mathcal{E}}$.

Let $S \rightarrow \mathfrak{M} \leftarrow \mathcal{M}$ still be a scheme over the coarse space \mathfrak{M} of a \mathbb{G}_m -gerbe \mathcal{M} . Then $\mathcal{S} := \mathcal{M} \times_{\mathfrak{M}} S$ is a \mathbb{G}_m -gerbe with coarse space S ; this \mathbb{G}_m -gerbe is neutral if and only if there is a Poincaré family \mathcal{E} on S .

3. VECTOR BUNDLES ON \mathbb{G}_m -GERBES

Definition 3.1. Let (\mathcal{M}, ι) be a \mathbb{G}_m -stack over k . A vector bundle \mathcal{V} on \mathcal{M} has weight $w \in \mathbb{Z}$ if the diagram

$$\begin{array}{ccc} \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\iota(\mathcal{E})} & \text{Aut}_{\mathcal{M}(S)}(\mathcal{E}) \\ (\cdot)^w \downarrow & & \downarrow \mathcal{V}(S) \\ \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\text{id}_{\mathcal{V}(\mathcal{E})}} & \text{Aut}_{\mathcal{O}_S}(\mathcal{V}(\mathcal{E})) \end{array}$$

commutes for every k -scheme S and every object \mathcal{E} of the groupoid $\mathcal{M}(S)$.

Example 3.2. The trivial vector bundle \mathcal{O}^n on any \mathbb{G}_m -stack \mathcal{M} has weight 0.

Examples 3.3. Given a point $P \in C(k)$, we denote by $\mathcal{E}_P^{\text{univ}}$ the restriction of the universal vector bundle $\mathcal{E}^{\text{univ}}$ on $C \times \mathcal{B}un_{r,L}$ to $\{P\} \times \mathcal{B}un_{r,L} \cong \mathcal{B}un_{r,L}$.

- i) $\mathcal{E}_P^{\text{univ}}$ is a vector bundle of weight 1 on $\mathcal{B}un_{r,L}$.
- ii) $(\mathcal{E}_P^{\text{univ}})^{\text{dual}}$ is a vector bundle of weight -1 on $\mathcal{B}un_{r,L}$.

Examples 3.4. We fix $r \in \mathbb{N}$, a line bundle L and a vector bundle F on C . Using the notation of the examples 1.11.ii and iii, we have:

- i) The vector bundle $\text{Hom}(F, \mathcal{E}^{\text{univ}})$ on $\mathcal{U}'_F \subseteq \mathcal{B}un_{r,L}$ has weight 1.
- ii) The vector bundle $\text{Hom}(\mathcal{E}^{\text{univ}}, F)$ on $\mathcal{U}''_F \subseteq \mathcal{B}un_{r,L}$ has weight -1 .

Example 3.5. The formalism of determinant line bundles [24] yields a line bundle \mathcal{L}_{det} on $\mathcal{B}un_{r,L}$ whose fibre over any point $[E]$ is $\det H^0(E) \otimes \det^{-1} H^1(E)$. This \mathcal{L}_{det} is a line bundle of weight $r(1-g) + \deg(L)$ on $\mathcal{B}un_{r,L}$ by Riemann-Roch.

We recall that vector bundles \mathcal{V} on a fixed algebraic stack \mathcal{M} over k form a category; a morphism $\mathcal{V} \rightarrow \mathcal{W}$ of such vector bundles can be described by a morphism $\mathcal{V}(\mathcal{E}) \rightarrow \mathcal{W}(\mathcal{E})$ of vector bundles on the k -scheme S for every object \mathcal{E} of $\mathcal{M}(S)$.

Lemma 3.6. *Let $\pi : \mathcal{M} \rightarrow \mathfrak{M}$ be the coarse space associated to a \mathbb{G}_m -gerbe \mathcal{M} over k . Sending V to $\mathcal{V} := \pi^*(V)$ defines an equivalence between the category of vector bundles V on \mathfrak{M} and the category of weight 0 vector bundles \mathcal{V} on \mathcal{M} .*

Proof. We claim that the inverse equivalence is given by the functor π_* from quasi-coherent $\mathcal{O}_{\mathcal{M}}$ -modules to quasi-coherent $\mathcal{O}_{\mathfrak{M}}$ -modules [26, Prop. 13.2.6 (iii)]. Thus we have to show that π_* sends weight 0 vector bundles \mathcal{V} on \mathcal{M} to vector bundles V on \mathfrak{M} and that the two adjunction morphisms $\pi^*\pi_*\mathcal{V} \rightarrow \mathcal{V}$ and $V \rightarrow \pi_*\pi^*V$ are isomorphisms. All this can be verified locally in the fppf-topology on \mathfrak{M} , so we may assume that the \mathbb{G}_m -gerbe \mathcal{M} is neutral; this special case is easy to check. \square

Example 3.7. If $\deg(L) = r(g-1)$, then the determinant line bundle \mathcal{L}_{det} on $\mathcal{B}un_{r,L}$ has weight 0; thus its restriction to $\mathcal{B}un_{r,L}^{\text{stab}}$ descends to $\mathfrak{B}un_{r,L}$ by lemma 3.6. In fact, \mathcal{L}_{det} descends to a line bundle $\mathfrak{L}_{\text{det}}$ on $\overline{\mathfrak{B}un}_{r,L}$ by [23, Prop. 4.2].

Corollary 3.8. *Let \mathcal{V} be a vector bundle of weight 0 on an irreducible \mathbb{G}_m -gerbe \mathcal{M} over k . Then \mathcal{V} is trivial on some open substack $\emptyset \neq \mathcal{U} \subseteq \mathcal{M}$.*

Proof. We have $\mathcal{V} \cong \pi^*(V)$ for a vector bundle V on \mathfrak{M} . [25, Prop. II.6.7] states that some open subspace $\emptyset \neq \mathfrak{U} \subseteq \mathfrak{M}$ is a scheme; hence V is trivial on some open subscheme $\emptyset \neq U \subseteq \mathfrak{U}$, and \mathcal{V} is trivial on $\mathcal{U} := \pi^{-1}(U)$. \square

Lemma 3.9. *Let \mathcal{M} be a \mathbb{G}_m -gerbe over k with coarse space $\pi : \mathcal{M} \rightarrow \mathfrak{M}$. This \mathbb{G}_m -gerbe is neutral if and only if there is a line bundle \mathcal{L} of weight 1 on \mathcal{M} .*

Proof. If \mathcal{M} is neutral, say $\mathcal{M} = \mathfrak{M} \times \mathrm{B}\mathbb{G}_m$, then pulling back the canonical line bundle L^{univ} of weight 1 on $\mathrm{B}\mathbb{G}_m$ yields a line bundle of weight 1 on \mathcal{M} .

Conversely, suppose that there is a line bundle \mathcal{L} of weight 1 on \mathcal{M} , and let $p : \mathrm{Tot}(\mathcal{L})^* \rightarrow \mathcal{M}$ be the complement of the zero section in its total space. We claim that p is a section for π , i. e. that $\pi \circ p : \mathrm{Tot}(\mathcal{L})^* \rightarrow \mathfrak{M}$ is a 1-isomorphism. This can be checked locally in the fppf-topology on \mathfrak{M} , so we may assume $\mathcal{M} = \mathfrak{M} \times \mathrm{B}\mathbb{G}_m$. Now \mathcal{L} and the pull-back of L^{univ} differ by tensoring with a weight 0 line bundle, so they are isomorphic locally in the fppf-topology on \mathfrak{M} by lemma 3.6; hence we may even assume that \mathcal{L} is the pull-back of L^{univ} . This special case is easy to check. \square

Lemma 3.10. *Let \mathcal{M} be an integral \mathbb{G}_m -gerbe over k with coarse space $\mathcal{M} \xrightarrow{\pi} \mathfrak{M}$.*

- i) There is a vector bundle \mathcal{V}_1 of weight 1 on some open substack $\emptyset \neq \mathcal{U}_1 \subseteq \mathcal{M}$.*
- ii) The generic fibre of the algebra bundle $\mathrm{End}(\mathcal{V}_1)$ on \mathcal{U}_1 descends to a central simple algebra A over the function field $k(\mathfrak{M})$.*
- iii) The Brauer class $\psi_{\mathcal{M}} := [A] \in \mathrm{Br}(k(\mathfrak{M}))$ of the central simple algebra A in ii does not depend on the choice of \mathcal{U}_1 and \mathcal{V}_1 made in i.*
- iv) Any two vector bundles $\mathcal{V}, \mathcal{V}'$ of the same rank n and weight w on open substacks $\emptyset \neq \mathcal{U}, \mathcal{U}' \subseteq \mathcal{M}$ are isomorphic on some open substack $\emptyset \neq \mathcal{U}'' \subseteq \mathcal{U} \cap \mathcal{U}'$.*
- v) There is a vector bundle of given rank n and given weight w on some open substack $\emptyset \neq \mathcal{U} \subseteq \mathcal{M}$ if and only if the index of $w \cdot \psi_{\mathcal{M}} \in \mathrm{Br}(k(\mathfrak{M}))$ divides n .*
- vi) There is a Poincaré family \mathcal{E} on some integral, generically finite scheme S of given degree n over \mathfrak{M} if and only if the index of $\psi_{\mathcal{M}}$ itself divides n .*

Proof. Replacing the base field k by the function field $k(\mathfrak{M})$ and \mathcal{M} by the \mathbb{G}_m -gerbe $\mathcal{M} \times_{\mathfrak{M}} \mathrm{Spec}(k(\mathfrak{M}))$ over it, we may assume $\mathfrak{M} = \mathrm{Spec}(k)$; then the only open substack $\emptyset \neq \mathcal{U} \subseteq \mathcal{M}$ is $\mathcal{U} = \mathcal{M}$ itself.

i) Due to Hilbert's Nullstellensatz, there is a finite field extension $k_1 \supseteq k$ with $\mathcal{M}(k_1) \neq \emptyset$. Then the \mathbb{G}_m -gerbe $\mathcal{M}_1 := \mathcal{M} \times_k \mathrm{Spec}(k_1)$ is neutral, so there is a line bundle \mathcal{L}_1 of weight 1 on \mathcal{M}_1 by lemma 3.9. Since the first projection $\mathrm{pr} : \mathcal{M}_1 \rightarrow \mathcal{M}$ is finite, flat and has weight 1, we get a vector bundle $\mathrm{pr}_* \mathcal{L}_1$ of weight 1 on \mathcal{M} .

ii) Let \mathcal{V}_1 be a vector bundle of weight 1 and rank n_1 on $\mathcal{U}_1 = \mathcal{M}$. The vector bundle $\mathrm{End}(\mathcal{V}_1)$ of weight 0 on \mathcal{M} and its multiplication descend to an algebra A of dimension n_1^2 over k by lemma 3.6. With k_1 , $\mathrm{pr} : \mathcal{M}_1 \rightarrow \mathcal{M}$ and \mathcal{L}_1 as in i,

$$A \otimes_k k_1 \cong \mathrm{End}(\mathrm{pr}^* \mathcal{V}_1) \cong \mathrm{End}(\mathrm{pr}^* \mathcal{V}_1 \otimes \mathcal{L}_1^{\mathrm{dual}})$$

is a full matrix algebra over k_1 because the weight 0 vector bundle $\mathrm{pr}^* \mathcal{V}_1 \otimes \mathcal{L}_1^{\mathrm{dual}}$ on \mathcal{M}_1 is trivial by corollary 3.8; this shows that A is central simple over k .

iii – v) If \mathcal{V} is a vector bundle of rank n and weight w on \mathcal{M} , then the weight 0 vector bundle $\mathrm{Hom}(\mathcal{V}_1^{\otimes w}, \mathcal{V})$ on \mathcal{M} and the right action of $\mathrm{End}(\mathcal{V}_1)^{\otimes w}$ on it descend to a right module M under the k -algebra $A^{\otimes w}$ with $\dim_k(M) = n_1^w \cdot n$ by lemma 3.6. This defines a Morita equivalence between vector bundles \mathcal{V} of weight w on \mathcal{M} and finitely generated right $A^{\otimes w}$ -modules M ; its inverse sends M to $\mathcal{V} := \pi^* M \otimes \mathcal{V}_1^{\otimes w}$ where the tensor product is taken over $\pi^* A^{\otimes w} \cong \mathrm{End}(\mathcal{V}_1^{\otimes w})$.

In particular, the category of finitely generated right A -modules is independent of \mathcal{V}_1 (up to equivalence); this implies iii. According to the theory of central simple algebras, a right $A^{\otimes w}$ -module M is determined up to isomorphism by $\dim_k(M)$,

and there is one with $\dim_k(M) = n_1^w \cdot n$ if and only if the index of $[A^{\otimes w}]$ divides n ; this proves iv and v.

vi) Here $S = \text{Spec}(K)$ for a finite field extension $K \supseteq k$. There is a Poincaré family \mathcal{E} on $S = \text{Spec}(K)$ if and only if the \mathbb{G}_m -gerbe $\mathcal{M}_K := \mathcal{M} \times_k \text{Spec}(K)$ is neutral, i. e. if and only if its Brauer class $\psi_{\mathcal{M}_K} \in \text{Br}(K)$ vanishes; the latter holds if and only if K is a splitting field for $\psi_{\mathcal{M}}$, because $\psi_{\mathcal{M}_K} = \psi_{\mathcal{M}} \otimes_k K$ in general. By the theory of central simple algebras again, $\psi_{\mathcal{M}} \in \text{Br}(k)$ has a splitting field of given degree n over k if and only if its index divides n . \square

Remark 3.11. If $\mathcal{M} = \mathcal{Bun}_{r,d}$, then $\psi_{\mathcal{M}}$ is precisely the Brauer class $\psi_{r,d}$ used by King and Schofield [21, Def. 3.3]. The above lemma is based on their arguments.

Remark 3.12. Much more information about vector bundles on \mathbb{G}_m -gerbes, in particular about *their* moduli spaces, can be found in M. Lieblich's work [28, 29].

Proposition 3.13. *Suppose that there is a line bundle \mathcal{L} of weight $w \in \mathbb{Z}$ on an open substack $\emptyset \neq \mathcal{U} \subseteq \mathcal{Bun}_{r,L}$ with $\deg(L) = 0$. Then r divides w .*

Proof. Considering $\mathcal{L} \otimes_k \bar{k}$ instead of \mathcal{L} , we may assume that k is algebraically closed. Then L is the r -th tensor power of a line bundle on C , tensoring with which defines an isomorphism $\mathcal{Bun}_{r,\mathcal{O}} \cong \mathcal{Bun}_{r,L}$. Hence we may also assume $L = \mathcal{O}$.

We denote by $[\mathcal{O}^r]$ the point on $\mathcal{Bun}_{r,\mathcal{O}}$ corresponding to the trivial bundle \mathcal{O}^r on C . Shrinking \mathcal{U} if necessary, we may suppose that \mathcal{U} is quasicompact; then we can extend \mathcal{L} to a neighbourhood of the point $[\mathcal{O}^r]$ as follows:

Choose a quasicompact open substack \mathcal{U}' of $\mathcal{Bun}_{r,\mathcal{O}}$ that contains \mathcal{U} and $[\mathcal{O}^r]$. [26, Cor. 15.5] allows us to extend \mathcal{L} to a coherent sheaf on \mathcal{U}' , more precisely to a coherent subsheaf $\mathcal{F} \subseteq j_*\mathcal{L}$ where $j : \mathcal{U} \hookrightarrow \mathcal{U}'$ is the open embedding. Then the double dual $\mathcal{L}' := \mathcal{F}^{**}$ is a reflexive coherent sheaf of rank one on the smooth stack \mathcal{U}' and hence a line bundle, cf. [6, Chap. VII, §4.2].

Now \mathcal{L}' is a line bundle of weight w because its restriction \mathcal{L} to \mathcal{U} is so. In particular, the scalars $\mathbb{G}_m \subseteq \text{GL}_r = \text{Aut}(\mathcal{O}^r)$ act with weight w on the fibre of \mathcal{L}' over $[\mathcal{O}^r]$. But any one-dimensional representation of GL_r factors through the determinant, so its weight w is always a multiple of r . \square

Corollary 3.14. *We fix a line bundle L of degree 0 on C , $r, n \geq 1$ and $w \in \mathbb{Z}$.*

i) *There is a vector bundle \mathcal{V} of rank n and weight w on some open substack $\emptyset \neq \mathcal{U} \subseteq \mathcal{Bun}_{r,L}$ if and only if r divides $w \cdot n$.*

ii) *There is a Poincaré family \mathcal{E} on some integral, generically finite scheme S of degree n over $\mathfrak{Bun}_{r,L}$ if and only if r divides n .*

Proof. If \mathcal{V} is a vector bundle of rank n and weight w on \mathcal{U} , then $\mathcal{L} := \det(\mathcal{V})$ is a line bundle of weight $w \cdot n$; the ‘only if’ in i thus follows from the previous proposition. According to lemma 3.10.v, this means that r divides w times the index of the Brauer class $w \cdot \psi_{\mathcal{Bun}_{r,L}} \in \text{Br}(k(\mathfrak{Bun}_{r,L}))$.

On the other hand, the vector bundle $\mathcal{E}_P^{\text{univ}}$ of rank r and weight 1 on $\mathcal{Bun}_{r,L}$ shows that the index of $\psi_{\mathcal{Bun}_{r,L}}$ divides r . Due to [2, 5.4], this implies that the index of $w \cdot \psi_{\mathcal{Bun}_{r,L}}$ divides r/w if w divides r ; together with the previous paragraph, this shows that the index of $w \cdot \psi_{\mathcal{Bun}_{r,L}}$ equals r/w if w divides r .

For general w , the Brauer classes $\text{hcf}(r, w) \cdot \psi_{\mathcal{Bun}_{r,L}}$ and $w \cdot \psi_{\mathcal{Bun}_{r,L}}$ have the same splitting fields because $w/\text{hcf}(r, w)$ is invertible modulo r ; hence the index $r/\text{hcf}(r, w)$ of the former is also the index of the latter. Using this, i and ii follow from lemma 3.10.v and vi. \square

4. GRASSMANNIAN BUNDLES

Given a vector bundle \mathcal{V} on a stack \mathcal{M} over k , we denote by $\mathrm{Gr}_j(\mathcal{V}) \rightarrow \mathcal{M}$ the associated Grassmannian bundle of j -dimensional linear subspaces for $j \leq \mathrm{rk}(\mathcal{V})$. Recall that this is a smooth representable 1-morphism of stacks over k whose fibre over the point of \mathcal{M} corresponding to an object E of $\mathcal{M}(k)$ is the usual Grassmannian scheme $\mathrm{Gr}_j(\mathcal{V}(E))$.

If (\mathcal{M}, ι) is a \mathbb{G}_m -stack over k and \mathcal{V} has some weight $w \in \mathbb{Z}$, then $\mathrm{Gr}_j(\mathcal{V})$ becomes a \mathbb{G}_m -stack over k as well, because all linear subspaces of $\mathcal{V}(E)$ are invariant under the \mathbb{G}_m -action given by ι . If \mathcal{M} is even a \mathbb{G}_m -gerbe over k , then $\mathrm{Gr}_j(\mathcal{V})$ also is.

Remark 4.1. There is a canonical 1-isomorphism $\mathrm{Gr}_j(\mathcal{V}) \xrightarrow{\sim} \mathrm{Gr}_{\mathrm{rk}(\mathcal{V})-j}(\mathcal{V}^{\mathrm{dual}})$ that sends each linear subspace of $\mathcal{V}(E)$ to its orthogonal complement in $\mathcal{V}(E)^{\mathrm{dual}}$.

Examples 4.2. We denote by $\mathcal{P}ar_{r,L}^{m/P}$ the moduli stack of rank r vector bundles E on C with $\det(E) \cong L$, endowed with a quasiparabolic structure of multiplicity m over the point $P \in C(k)$ in the sense of [30]. We recall that such a quasiparabolic structure can be given by a coherent subsheaf $E' \subseteq E$ for which E/E' is isomorphic to the skyscraper sheaf \mathcal{O}_P^m .

- i) $\mathcal{P}ar_{r,L}^{m/P}$ is canonically 1-isomorphic to $\mathrm{Gr}_m((\mathcal{E}_P^{\mathrm{univ}})^{\mathrm{dual}})$ over $\mathcal{B}un_{r,L}$.
- ii) $\mathcal{P}ar_{r,L}^{m/P}$ is also 1-isomorphic to $\mathrm{Gr}_m(\mathcal{E}_P^{\mathrm{univ}})$ over $\mathcal{B}un_{r,L(-mP)}$.

Proof. i) The quasiparabolic vector bundle $E^\bullet = (E' \subseteq E)$ is given by the vector bundle E together with a dimension m quotient of the fibre E_P .

ii) E^\bullet is also given by the vector bundle E' together with a dimension m vector subspace in the fibre $E'(P)_P$ of the twisted bundle $E'(P)$. Choosing once and for all a trivialisation at P for the line bundle $\mathcal{O}_C(P)$ identifies $E'(P)_P$ and E'_P . \square

In analogy with [21], the proof of our main result will use not only the above Hecke correspondence, but also the following slightly more involved comparison of Grassmannian bundles over different moduli stacks $\mathcal{B}un_{r,L}$:

Examples 4.3. Given $j, r_1, r_2, r_3 \geq 1$ with $r_1 + r_3 = jr_2$ and line bundles L_1, L_2, L_3 on C with $L_1 \otimes L_3 \cong L_2^{\otimes j}$, let \mathcal{M} be the moduli stack of all exact sequences

$$(10) \quad 0 \longrightarrow E_1 \xrightarrow{i} E_2 \otimes_k V \xrightarrow{p} E_3 \longrightarrow 0$$

in which V is a vector space over k and E_1, E_2, E_3 are vector bundles on C with $\dim(V) = j$ and $\mathrm{rk}(E_\nu) = r_\nu$, $\det(E_\nu) \cong L_\nu$ for $\nu = 1, 2, 3$. More precisely, $\mathcal{M}(S)$ is the following groupoid for each k -scheme S :

- An object consists of a rank j vector bundle \mathcal{V} on S , an object \mathcal{E}_ν of $\mathcal{B}un_{r_\nu, L_\nu}(S)$ for $\nu = 1, 2, 3$ and an exact sequence of bundles on $C \times_k S$

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \otimes_{\mathcal{O}_S} \mathcal{V} \longrightarrow \mathcal{E}_3 \longrightarrow 0.$$

- A morphism consists of four vector bundle isomorphisms $\lambda : \mathcal{V} \rightarrow \mathcal{V}'$ and $\varphi_\nu : \mathcal{E}_\nu \rightarrow \mathcal{E}'_\nu$, $\nu = 1, 2, 3$, such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_2 \otimes_{\mathcal{O}_S} \mathcal{V} & \longrightarrow & \mathcal{E}_3 \longrightarrow 0 \\ & & \downarrow \varphi_1 & & \downarrow \varphi_2 \otimes \lambda & & \downarrow \varphi_3 \\ 0 & \longrightarrow & \mathcal{E}'_1 & \longrightarrow & \mathcal{E}'_2 \otimes_{\mathcal{O}_S} \mathcal{V}' & \longrightarrow & \mathcal{E}'_3 \longrightarrow 0. \end{array}$$

\mathcal{M} is algebraic over k because the forgetful 1-morphism $\mathcal{M} \rightarrow \mathcal{Bun}_{r_2, L_2} \times \mathcal{Vect}_j$ is representable (by suitable open subschemes of relative Quot-schemes) and of finite type. We have a strictly commutative diagram of canonical 1-morphisms

$$\begin{array}{ccccc}
\mathrm{Gr}_j(\mathrm{Hom}(\mathcal{E}_1^{\mathrm{univ}}, \mathcal{E}_2^{\mathrm{univ}})) & \xleftarrow{\Gamma'} & \mathcal{M}' \subseteq \mathcal{M} \supseteq \mathcal{M}'' & \xrightarrow{\Gamma''} & \mathrm{Gr}_j(\mathrm{Hom}(\mathcal{E}_2^{\mathrm{univ}}, \mathcal{E}_3^{\mathrm{univ}})) \\
\downarrow & \nearrow \Phi_{1,2} & \downarrow & \searrow \Phi_{2,3} & \downarrow \\
\mathcal{U}_{1,2} \subseteq \mathcal{Bun}_{r_1, L_1} \times \mathcal{Bun}_{r_2, L_2} & & \mathcal{Bun}_{r_2, L_2} \times \mathcal{Bun}_{r_3, L_3} \supseteq \mathcal{U}_{2,3} & & \\
& \searrow & \downarrow \Phi_2 & \nearrow & \\
& & \mathcal{Bun}_{r_2, L_2} & &
\end{array}$$

in which $\Phi_{1,2}$, Φ_2 , $\Phi_{2,3}$ are forgetful morphisms,

- $\mathcal{U}_{\nu, \nu+1} \subseteq \mathcal{Bun}_{r_\nu, L_\nu} \times \mathcal{Bun}_{r_{\nu+1}, L_{\nu+1}}$ is the open substack defined by the condition $\mathrm{Ext}^1(E_\nu, E_{\nu+1}) = 0$,
- $\mathcal{M}'' \subseteq \Phi_{2,3}^{-1}(\mathcal{U}_{2,3})$ is the open substack where $p_* : V \rightarrow \mathrm{Hom}(E_2, E_3)$ is injective in (10), and Γ'' sends such a sequence to the image of p_* ,
- $\mathcal{M}' \subseteq \Phi_{1,2}^{-1}(\mathcal{U}_{1,2})$ is the open substack where $i^* : V^{\mathrm{dual}} \rightarrow \mathrm{Hom}(E_1, E_2)$ is injective in (10), and Γ' sends such a sequence to the image of i^* ,

and the remaining four unlabelled arrows are canonical projections. We claim:

- i) Γ' and Γ'' are open immersions.
- ii) If $\deg(L_{\nu+1})/r_{\nu+1} - \deg(L_\nu)/r_\nu > g-1$ for $\nu = 1, 2$ and j divides r_ν , $\deg(L_\nu)$ for $\nu = 1, 3$, then $\mathcal{M}' \cap \mathcal{M}''$ contains a k -rational point.

Proof. i) An open substack of $\mathrm{Gr}_j(\mathrm{Hom}(\mathcal{E}_2^{\mathrm{univ}}, \mathcal{E}_3^{\mathrm{univ}}))$ parameterises all linear subspaces $W \subseteq \mathrm{Hom}(E_2, E_3)$ for which the evaluation map $\epsilon_W : E_2 \otimes_k W \rightarrow E_3$ is surjective. Γ'' is in fact a 1-isomorphism over $\mathcal{U}_{2,3}$ onto that open substack; its inverse sends such a linear subspace W to the sequence

$$0 \longrightarrow \ker(\epsilon_W) \longrightarrow E_2 \otimes_k W \xrightarrow{\epsilon_W} E_3 \longrightarrow 0.$$

Hence Γ'' is an open immersion; considering $E_{4-\nu}^{\mathrm{dual}}$ and V^{dual} instead of E_ν and V , it follows that Γ' is an open immersion as well.

ii) Due to i and lemma 1.5, it suffices to show $\mathcal{M}' \cap \mathcal{M}'' \neq \emptyset$. Thus we may assume that $k = \bar{k}$ is algebraically closed.

In this case, we can choose a line bundle \tilde{L}_1 on C with $\tilde{L}_1^{\otimes j} \cong L_1$; we define $\tilde{L}_3 := L_2 \otimes \tilde{L}_1^{\mathrm{dual}}$. Instead of j , r_1, r_2, r_3 and L_1, L_2, L_3 , we consider $\tilde{j} := 1$, $r_1/\tilde{j}, r_2, r_3/\tilde{j}$ and $\tilde{L}_1, L_2, \tilde{L}_3$. We denote the resulting moduli stack by $\tilde{\mathcal{M}}$ and the analogous open substacks by $\tilde{\mathcal{M}}' \subseteq \tilde{\mathcal{M}} \supseteq \tilde{\mathcal{M}}''$. If $\tilde{\mathcal{M}}' \cap \tilde{\mathcal{M}}''$ has a k -rational point $0 \rightarrow \tilde{E}_1 \rightarrow E_2 \otimes_k \tilde{V} \rightarrow \tilde{E}_3 \rightarrow 0$, then taking the direct sum of j copies yields a k -rational point of $\mathcal{M}' \cap \mathcal{M}''$. Hence it suffices to treat the case $j = 1$.

In this case $j = 1$, proposition 1.12 implies that the image of Γ'' described above is nonempty; hence $\mathcal{M}'' \neq \emptyset$. The dual argument shows $\mathcal{M}' \neq \emptyset$. But the forgetful 1-morphism $\mathcal{M} \rightarrow \mathcal{Bun}_{r_1, L_1} \times \mathcal{Vect}_1 \times \mathcal{Bun}_{r_3, L_3}$ is smooth with all fibers irreducible due to lemma 1.10; hence \mathcal{M} is irreducible. This proves $\mathcal{M}' \cap \mathcal{M}'' \neq \emptyset$ here. \square

Corollary 4.4. *In the above notation, let the vector bundle E_2 on C belong to a k -rational point (10) of $\mathcal{M}' \cap \mathcal{M}''$. Then the Grassmannian $\mathrm{Gr}_j(\mathrm{Hom}(E_2, \mathcal{E}_3^{\mathrm{univ}}))$ over the open substack $\emptyset \neq (\mathcal{U}_{2,3})_{E_2} \subseteq \mathcal{Bun}_{r_3, L_3}$ is birational to the Grassmannian $\mathrm{Gr}_j(\mathrm{Hom}(\mathcal{E}_1^{\mathrm{univ}}, E_2))$ over the open substack $\emptyset \neq (\mathcal{U}_{1,2})_{E_2} \subseteq \mathcal{Bun}_{r_1, L_1}$.*

Proof. Both Grassmannian bundles contain as a nonempty open substack the fibre of the forgetful 1-morphism $\Phi_2 : \mathcal{M}' \cap \mathcal{M}'' \rightarrow \mathcal{Bun}_{r_2, L_2}$ over E_2 . \square

Lemma 4.5. *Suppose that \mathcal{V} and \mathcal{W} are two vector bundles of the same weight w on an irreducible \mathbb{G}_m -gerbe \mathcal{M} over k , and let $j \leq \text{rk}(\mathcal{V})$ be given.*

- i) $\text{Gr}_j(\mathcal{V} \oplus \mathcal{W})$ is birational to $\mathbb{A}^{j \cdot \text{rk}(\mathcal{W})} \times \text{Gr}_j(\mathcal{V})$.
- ii) $\text{Gr}_{j+\text{rk}(\mathcal{W})}(\mathcal{V} \oplus \mathcal{W})$ is birational to $\mathbb{A}^s \times \text{Gr}_j(\mathcal{V})$ with $s := (\text{rk}(\mathcal{V}) - j) \cdot \text{rk}(\mathcal{W})$.

Proof. i) Let $\emptyset \neq \mathcal{U} \subseteq \text{Gr}_j(\mathcal{V} \oplus \mathcal{W})$ be the open substack that parameterises all the j -dimensional vector subspaces of $\mathcal{V}(E) \oplus \mathcal{W}(E)$ whose image S in $\mathcal{V}(E)$ still has dimension j . Each of these subspaces is the graph of a unique linear map $S \rightarrow \mathcal{W}(E)$. Denoting by $\mathcal{S}^{\text{univ}} \subseteq p^*\mathcal{V}$ the universal subbundle on $p : \text{Gr}_j(\mathcal{V}) \rightarrow \mathcal{M}$, this defines a canonical 1-isomorphism between \mathcal{U} and the total space of the vector bundle $\text{Hom}(\mathcal{S}^{\text{univ}}, p^*\mathcal{W})$ on $\text{Gr}_j(\mathcal{V})$. The latter has weight 0 because $\mathcal{S}^{\text{univ}}$ and $p^*\mathcal{W}$ both have weight w ; hence corollary 3.8 applies to it.

- ii) follows from i by applying remark 4.1 twice. \square

Proposition 4.6. *Let L be a line bundle of degree 0 on C . Let \mathcal{V} be a vector bundle of rank n and weight w on an open substack $\emptyset \neq \mathcal{U} \subseteq \mathcal{Bun}_{r, L}$, and let $j \leq n$.*

- i) If r divides $w \cdot j$, then $\text{Gr}_j(\mathcal{V})$ is birational to $\mathbb{A}^{j(n-j)} \times \mathcal{Bun}_{r, L}$.
- ii) If $w \in \{0, \pm 1\}$, then $\text{Gr}_j(\mathcal{V})$ is birational to $\mathbb{A}^s \times \text{Gr}_m(\mathcal{E}_P^{\text{univ}})$ for some s , where $m \in \{0, 1, \dots, r-1\}$ is the remainder of $w \cdot j$ modulo r .
- iii) If r and $w \cdot j$ are coprime, then $\text{Gr}_j(\mathcal{V})$ is stably birational to $\text{Gr}_1(\mathcal{E}_P^{\text{univ}})$.

Proof. Note that r divides $w \cdot n$ due to corollary 3.14.i.

i) Corollary 3.14.i also implies that there are weight w vector bundles \mathcal{V}' and \mathcal{W} of rank j and $n-j$ on some open substack $\emptyset \neq \mathcal{U}' \subseteq \mathcal{Bun}_{r, L}$. According to lemma 3.10.iv, we have $\mathcal{V} \cong \mathcal{V}' \oplus \mathcal{W}$ on some nonempty open substack of $\mathcal{U} \cap \mathcal{U}'$; hence $\text{Gr}_j(\mathcal{V})$ is by lemma 4.5.i birational to $\mathbb{A}^{j(n-j)} \times \text{Gr}_j(\mathcal{V}') \cong \mathbb{A}^{j(n-j)} \times \mathcal{U}'$.

ii) The case $w = 0$ follows from i. Using remark 4.1, it suffices to treat the case $w = 1$. Here lemma 3.10.iv allows us to assume $\mathcal{V} = (\mathcal{E}_P^{\text{univ}})^{n/r}$. Then $\text{Gr}_j(\mathcal{V})$ is by lemma 4.5.ii birational to $\mathbb{A}^{s'} \times \text{Gr}_m(\mathcal{V}')$ for some s' with $\mathcal{V}' := (\mathcal{E}_P^{\text{univ}})^{(n-j+m)/r}$. Now the claim follows from lemma 4.5.i.

iii) Let the algebraic stack \mathcal{M} be the fibred product of $p : \text{Gr}_j(\mathcal{V}) \rightarrow \mathcal{U}$ and $q : \text{Gr}_1(\mathcal{E}_P^{\text{univ}}) \rightarrow \mathcal{U}$. Denoting by $\mathcal{S}^{\text{univ}} \subseteq p^*\mathcal{V}$ the universal subbundle, $\det(\mathcal{S}^{\text{univ}})$ is a line bundle of weight $w \cdot j$ on $\text{Gr}_j(\mathcal{V})$, whereas $p^*\det(\mathcal{E}_P^{\text{univ}})$ is a line bundle of weight r ; since these weights are coprime, there also is a line bundle \mathcal{L} of weight 1 on $\text{Gr}_j(\mathcal{V})$. Applying corollary 3.8 to $\mathcal{L}^{\text{dual}} \otimes p^*\mathcal{E}_P^{\text{univ}}$, we see that the Grassmannian bundle $\mathcal{M} \cong \text{Gr}_1(p^*\mathcal{E}_P^{\text{univ}})$ over $\text{Gr}_j(\mathcal{V})$ is birational to $\mathbb{A}^{r-1} \times \text{Gr}_j(\mathcal{V})$.

On the other hand, the universal subbundle of $q^*\mathcal{E}_P^{\text{univ}}$ is a line bundle of weight 1 on $\text{Gr}_1(\mathcal{E}_P^{\text{univ}})$; applying 3.8 as above, we see that the Grassmannian bundle $\mathcal{M} \cong \text{Gr}_j(q^*\mathcal{V})$ over $\text{Gr}_1(\mathcal{E}_P^{\text{univ}})$ is also birational to $\mathbb{A}^{j(n-j)} \times \text{Gr}_1(\mathcal{E}_P^{\text{univ}})$. \square

Remarks 4.7. i) Amitsur's conjecture [1, p. 40] about generic splitting fields of central simple algebras would allow to remove 'stably' in iii if $j = 1$ and $n = r$. (Indeed, both Grassmannians contain *neutral* \mathbb{G}_m -gerbes as open substacks here, and their coarse spaces have isomorphic function fields according to the conjecture.)

ii) If $w \notin \{0, \pm 1\}$ and r neither divides nor is coprime to $w \cdot j$, then the index of the Brauer class $\psi_{\text{Gr}_j(\mathcal{V})}$ need no longer coincide with its order in the Brauer group, according to [37] for $j = 1$ and to [4], [39] for general j . Thus there seems to be no direct generalisation of iii to this case.

5. PROOF OF THE MAIN THEOREM

Theorem 5.1. *Let L be a line bundle of degree d on C . We consider an open substack $\emptyset \neq \mathcal{U}_1 \subseteq \mathcal{Bun}_{r,L}$ and a tower of Grassmannian bundles*

$$\mathcal{M} = \mathrm{Gr}_{j_T}(\mathcal{V}_T) \longrightarrow \mathcal{U}_T \subseteq \mathrm{Gr}_{j_{T-1}}(\mathcal{V}_{T-1}) \longrightarrow \mathcal{U}_{T-1} \dots \mathrm{Gr}_{j_1}(\mathcal{V}_1) \longrightarrow \mathcal{U}_1 \subseteq \mathcal{Bun}_{r,L}$$

where \mathcal{V}_t is a vector bundle of weight $w_t \in \{0, \pm 1\}$ and rank $n_t \geq j_t$ on \mathcal{U}_t for each t , and $\emptyset \neq \mathcal{U}_{t+1} \subseteq \mathrm{Gr}_{j_t}(\mathcal{V}_t)$ is again an open substack.

Let h denote the highest common factor of $r, d, w_1 \cdot j_1, \dots, w_T \cdot j_T$. Then \mathcal{M} is birational to $\mathbb{A}^s \times \mathcal{Bun}_{h,L_0}$ for the degree 0 line bundle $L_0 := L(-dP)$ on C with

$$s = \dim \mathcal{M} - \dim \mathcal{Bun}_{h,L_0} = (g-1)(r^2 - h^2) + \sum_t j_t(n_t - j_t).$$

Proof. i) We start with the case $h = r$, i.e. r divides d and $w_1 j_1, \dots, w_T j_T$. We have a 1-isomorphism $\mathcal{Bun}_{r,L_0} \rightarrow \mathcal{Bun}_{r,L}$ defined by sending each vector bundle E on C to its twist $E(\frac{d}{r}P)$; hence we may assume $d = 0$ without loss of generality. Then $\mathrm{Gr}_{j_1}(\mathcal{V}_1)$ is birational to $\mathbb{A}^{j_1(n_1 - j_1)} \times \mathcal{Bun}_{r,L_0}$ due to proposition 4.6.i; replacing k by the function field of $\mathbb{A}^{j_1(n_1 - j_1)}$ and iterating, the theorem follows here.

For the remaining case $h < r$, we use induction on r/h . We first consider the case that r divides d and denote by $t \leq T$ the smallest index for which r does not divide $w_t j_t$. Then \mathcal{U}_t is birational to $\mathbb{A}^{s_1} \times \mathcal{Bun}_{r,L_0}$ for some s_1 by the previous paragraph; replacing k by the function field of \mathbb{A}^{s_1} , we may assume $s_1 = 0$ and $t = 1$, i.e. $\mathcal{U}_1 \subseteq \mathcal{Bun}_{r,L_0}$ and r does not divide $w_1 \cdot j_1$. Here proposition 4.6.ii makes \mathcal{U}_2 birational to $\mathbb{A}^{s_2} \times \mathrm{Gr}_m(\mathcal{E}_P^{\mathrm{univ}})$ for some s_2 , where $m \in \{1, \dots, r-1\}$ is the remainder of $w_1 \cdot j_1$ modulo r . Again, we can achieve $s_2 = 0$ by changing k . But then \mathcal{U}_2 is also birational to $\mathrm{Gr}_m((\mathcal{E}_P^{\mathrm{univ}})^{\mathrm{dual}}) \rightarrow \mathcal{Bun}_{r,L_0(mP)}$ by the examples 4.2. Thus we may assume for the induction step that r does not divide d .

Let j be the highest common factor of r and d ; then $r/j > 1$. According to elementary number theory, there is a unique integer r_2 with $r/j < r_2 < 2r/j$ such that $r_2 d \equiv j \pmod{r}$; we then define $d_2 \in \mathbb{Z}$ by the equation

$$(11) \quad (1-g)r_2 r + r_2 d - r d_2 = j$$

and put $L_2 := \mathcal{O}_C(d_2 P)$. Now we claim that the hypotheses of 4.3.ii are satisfied with $r_1 := jr_2 - r < r$, $r_3 := r$ and $L_1 := L^{\mathrm{dual}}(jd_2 P)$, $L_3 := L$:

Only the two inequalities in 4.3.ii are not obvious, but $d/r - d_2/r_2 > g-1$ follows from (11), and then $d_2/r_2 - d_1/r_1 > g-1$ for $d_1 := \deg(L_1)$ follows from the computation

$$r_1 d_2 - r_2 d_1 = (jr_2 - r)d_2 - r_2(jd_2 - d) = r_2 d - r d_2 > (g-1)r_2 r > (g-1)r_1 r_2.$$

Hence corollary 4.4 applies for some vector bundle E_2 on C of rank r_2 and determinant L_2 . It yields that $\mathrm{Gr}_j(\mathrm{Hom}(\mathcal{E}_1^{\mathrm{univ}}, E_2))$ over $(\mathcal{U}_{1,2})_{E_2} \subseteq \mathcal{Bun}_{r_1, L_1}$ is birational to Gr_j of the vector bundle $\mathrm{Hom}(E_2, \mathcal{E}^{\mathrm{univ}})$ on $(\mathcal{U}_{2,3})_{E_2} \subseteq \mathcal{Bun}_{r,L}$ whose rank is j by (11); thus it is birational to $\mathcal{Bun}_{r,L}$ itself. Example 2.7 shows that $\mathcal{Bun}_{r_1, L_1} \cong \mathcal{Bun}_{r_1, L(-jd_2 P)}$; now we can apply the induction hypothesis, observing that $r_1 < r$ and $\mathrm{hcf}(r_1, d_1, j) = j = \mathrm{hcf}(r, d)$. \square

Keeping the notation of theorem 5.1 for a while, let $\emptyset \neq \mathcal{U} \subseteq \mathcal{M}$ be an open substack which is a \mathbb{G}_m -gerbe, and let $\pi : \mathcal{U} \rightarrow \mathfrak{U}$ be its coarse space.

Corollary 5.2. i) \mathfrak{U} is birational to $\mathbb{A}^s \times \mathfrak{Bun}_{h,L_0}$ over k .
ii) If $h = 1$, then \mathfrak{U} is rational over k .

iii) If $g = h = 2$ and $\text{char}(k) = 0$, then \mathfrak{U} is rational over k .

Proof. i) is a consequence of theorem 5.1 and lemma 2.11.iii.

ii) follows from i, since $\mathfrak{Bun}_{1,L_0} \cong \text{Spec}(k)$.

iii) Using i, it suffices to show $\overline{\mathfrak{Bun}}_{2,L_0} \cong \mathbb{P}^3$, or equivalently $\overline{\mathfrak{Bun}}_{2,L_0(2P)} \cong \mathbb{P}^3$. Since C , L and P are defined over a finitely generated extension field of \mathbb{Q} , we may assume $k \subseteq \mathbb{C}$. Example 3.7 yields a determinant line bundle \mathcal{L}_{\det} on $\overline{\mathfrak{Bun}}_{2,L_0(2P)}$; we claim that its dual is very ample and that the resulting projective embedding is an isomorphism onto \mathbb{P}^3 . To check that, we may replace k by its extension field \mathbb{C} ; then the claim holds by [31, §7] and [3, Prop. 2.5]. \square

Corollary 5.3. *With \mathfrak{U} and h as above, the following are equivalent for $n \in \mathbb{N}$:*

- i) *There is an integral, generically finite scheme S of degree n over \mathfrak{U} that admits a Poincaré family \mathcal{E} on S .*
- ii) *h divides n .*

In particular, there is no Poincaré family on any open subspace of \mathfrak{U} if $h \neq 1$.

Proof. Combine theorem 5.1 with corollary 3.14.ii. \square

Corollary 5.4. *Let \mathcal{M} and h be as in theorem 5.1; similarly for \mathcal{M}' and h' . If $h \neq h'$, then the stacks \mathcal{M} and \mathcal{M}' are not stably birational.*

Proof. Suppose that $\mathbb{A}^s \times \mathcal{M}$ and $\mathbb{A}^{s'} \times \mathcal{M}'$ have a common open substack $\mathcal{U} \neq \emptyset$. We may assume that \mathcal{U} is a \mathbb{G}_m -gerbe; let \mathfrak{U} be its coarse space. Due to the previous corollary, h and h' both coincide with the minimal degree of integral, generically finite schemes S over \mathfrak{U} that admit a Poincaré family; hence $h = h'$. \square

Remark 5.5. Nevertheless, coarse spaces for \mathcal{M} and \mathcal{M}' might be stably birational for some $h \neq h'$, as is the case for $\text{char}(k) = 0$, $g = 2$ and $h = 1$, $h' = 2$ by corollary 5.2. All that 5.4 says is that an eventual proof of stable birationality for the coarse spaces has to contain arguments that do not carry over to the stacks.

Corollary 5.6. *Let \mathcal{M} and h be as above; let $n \in \mathbb{N}$ and $w \in \mathbb{Z}$ be given. Then there exists an open substack $\emptyset \neq \mathcal{U} \subseteq \mathcal{M}$ and a vector bundle \mathcal{V} of rank n and weight w on \mathcal{U} if and only if h divides $w \cdot n$.*

Proof. This follows directly from theorem 5.1, lemma 2.8 and corollary 3.14.i. \square

Example 5.7. Let $D \subset C(k)$ be a finite set of points on our curve C . We consider vector bundles E on C with fixed rank r and degree d , endowed with a quasi-parabolic structure with fixed multiplicities $m_1(P), \dots, m_l(P)$ at each point $P \in D$. Let $\mathfrak{Par}_{r,d}^{m/D}$ be their coarse moduli scheme for some given weights $\alpha_1, \dots, \alpha_l$, as introduced in [30]. Then the above implies in particular:

- i) All fibers of $\det : \mathfrak{Par}_{r,d}^{m/D} \rightarrow \text{Pic}^d(C)$ are rational if the highest common factor h of r , d and all multiplicities $m_l(P)$ is 1. (This generalises [5].)
- ii) There is a Poincaré family on some nonempty open subscheme of $\mathfrak{Par}_{r,d}^{m/D}$ if and only if $h = 1$.

Example 5.8. Concerning the coarse moduli scheme $\mathfrak{Bun}_{r,L}$ of vector bundles with fixed determinant L of degree d , the above reproves the rationality theorem of King and Schofield [21] and the non-existence theorem for Poincaré families due to Ramanan [34]. Moreover, we get the following additional information:

- i) There is an integral, generically finite scheme of degree n over $\mathfrak{Bun}_{r,L}$ with a Poincaré family if and only if $\text{hcf}(r, d)$ divides n .
 - ii) For a fixed vector bundle F on C , the minimal dimension of $\dim_k \text{Hom}(F, E)$ for rank r , determinant L vector bundles E on C is a multiple of $\text{hcf}(r, d)$.
- (ii follows from 5.6 and example 1.11.ii. For F generic, it also follows from the theorem of Hirschowitz [17] that $\text{Hom}(F, E) = 0$ or $\text{Ext}^1(F, E) = 0$ for E, F generic.)

Example 5.9. Given a closed subscheme $\emptyset \neq D \subsetneq C$, we consider rank r vector bundles E on C with $\det(E) \cong L$, endowed with a *level structure* at D , i. e. with a nonzero morphism $E \rightarrow \mathcal{O}_D^r$. Let $\mathfrak{Lcv}_{r,L}^D$ be the coarse moduli scheme of such E with level structure which are stable in the sense of [38]. Then $\mathfrak{Lcv}_{r,L}^D$ is rational.

Proof. Let \mathcal{V} be the vector bundle of weight -1 on $\mathcal{Bun}_{r,L}$ whose fibre at any point $[E]$ is $\text{Hom}_{\mathcal{O}_C}(E, \mathcal{O}_D^r)$. Then $\mathfrak{Lcv}_{r,L}^D$ is the coarse space associated to an open substack of $\text{Gr}_1(\mathcal{V})$; hence it is rational by corollary 5.2.ii above. \square

More generally, the above results can be applied in a similar way to *stable pairs* on C in the sense of Huybrechts and Lehn [19], and to other sorts of *augmented bundles* on C in the sense of [7]; we mention:

Example 5.10. A *coherent system* (E, V) of type (r, d, n) consists of a rank r , degree d vector bundle E on C and an n -dimensional vector subspace $V \subseteq H^0(E)$, cf. [27], [33] or [22]. Assuming $k = \mathbb{C}$, $n < r$, $0 < d$ and $n \leq d + (r - n)(g - 1)$, let $\mathfrak{Coh}_{r,d}^n$ be the coarse moduli scheme of such coherent systems which are α -stable for the largest possible value $\alpha = d/(r - n) - \epsilon$ of the stability parameter $\alpha \in \mathbb{R}$ [7, §3]. Then $\mathfrak{Coh}_{r,d}^n$ is birational to $\mathbb{A}^s \times \mathfrak{Bun}_{h,0}$ for $h := \text{hcf}(r, d, n)$ and some $s \in \mathbb{N}$.

Proof. Let \mathcal{V} be the vector bundle of weight -1 on $\mathcal{Bun}_{r-n,d}^{\text{stab}}$ whose fibre at any point $[E']$ is $\text{Ext}^1(E', \mathcal{O}_C)$; note that $\text{Hom}(E', \mathcal{O}_C) = 0$ due to the stability of E' , so \mathcal{V} is indeed a vector bundle of rank $d + (r - n)(g - 1) \geq n$. The coarse space of the Grassmannian $\text{Gr}_n(\mathcal{V})$ over $\mathcal{Bun}_{r-n,d}^{\text{stab}}$ is birational to $\mathfrak{Coh}_{r,d}^n$ according to the proof of [8, Thm. 4.5]; now apply corollary 5.2.i over the function field of $\text{Pic}^d(C)$. \square

Example 5.11. Let $\mathfrak{BN}_{r,d}^n \subseteq \mathfrak{Bun}_{r,d}$ be the *Brill-Noether locus* of vector bundles E on C with $\dim_k H^0(E) \geq n$. Assume $n = 1$ or that C is a Petri curve and $n \in \{2, 3\}$. If $k = \mathbb{C}$, $n < r$ and $0 < d < r(g - 1) + n$, then $\mathfrak{BN}_{r,d}^n$ is birational to $\mathbb{A}^s \times \mathfrak{Bun}_{h,0}$ for $h := \text{hcf}(r, d, n)$ and some $s \in \mathbb{N}$ (or empty if $n = 3$, $r = 4$, $d = 1$ and $g = 2$).

Proof. Here $\mathfrak{BN}_{r,d}^n$ is birational to $\mathfrak{Coh}_{r,d}^n$, according to [9, Section 11.2]. \square

There seems to be no easy generalisation of theorem 5.1 to weights $w \notin \{0, \pm 1\}$, mainly because an analogue of the Hecke correspondence 4.2 is missing; cf. also remark 4.7. But at least, we have the following:

Corollary 5.12. *Let L be a line bundle of degree d on C , let \mathcal{V} be a vector bundle of weight w on an open substack $\emptyset \neq \mathcal{U} \subseteq \mathcal{Bun}_{r,L}$, and let $j \leq \text{rk}(\mathcal{V})$. If r, d and $w \cdot j$ have no common factor $\neq \pm 1$, then $\text{Gr}_j(\mathcal{V})$ is stably birational to $\mathcal{Bun}_{1,L} \cong \text{BG}_m$.*

Proof. Replacing r by $\text{hcf}(r, d)$ and k by a rational function field over k , theorem 5.1 allows us to assume $d = 0$ without loss of generality. This case follows from proposition 4.6.iii and theorem 5.1 again. \square

Example 5.13. Over $k = \mathbb{C}$, let $\rho : \mathrm{GL}(r) \rightarrow \mathrm{GL}(V)$ be an algebraic representation that maps the center $\mathbb{G}_m \subseteq \mathrm{GL}(r)$ to the center $\mathbb{G}_m \subseteq \mathrm{GL}(V)$ with weight $w \in \mathbb{Z}$. Let M be a line bundle on C . We consider vector bundles E of rank r and degree d on C which are *decorated* in the sense of A. Schmitt [36], i. e. endowed with an element of $\mathbb{P}\mathrm{Hom}(E_\rho, M)$ where E_ρ is the vector bundle on C with fibre V associated to the $\mathrm{GL}(r)$ -bundle given by E .

We fix a stability parameter $\delta > 0$ and choose M of sufficiently large degree such that the resulting coarse moduli scheme $\mathfrak{D}\mathfrak{e}\mathfrak{c}_{r,d}^M$ of δ -stable decorated vector bundles is irreducible by [36, Thm. 3.5]. If r, d and w have no common factor $\neq \pm 1$, then the generic fibre of $\det : \mathfrak{D}\mathfrak{e}\mathfrak{c}_{r,d}^M \rightarrow \mathrm{Pic}^d(C)$ is stably rational.

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