

# **A NOTE ON THE COMPUTATION OF SYMMETRIC POWERS OF HYPERBOLIC FORMS AND OF TRACE FORMS ON SYMBOL ALGEBRAS**

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## **Abstract**

Let  $K$  be a field with characteristic different from 2 and let  $S$  be a symbol algebra over  $K$ . We compute the symmetric powers of hyperbolic quadratic forms over  $K$ . Also, we compute the symmetric powers of the quadratic trace form of  $S$ . In both cases, we apply a generalized form of the Vandermonde convolution in the course of the computations.

## **1. Introduction**

We assume throughout this paper that  $K$  is a field with characteristic different from 2. In [3], we computed the exterior powers of hyperbolic forms over  $K$  and used these results to compute the exterior powers of the quadratic trace form of symbol algebras over  $K$ . In this paper, we are interested in finding corresponding results for the symmetric powers of such forms.

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In [9], McGarraghy investigated symmetric powers of classes of symmetric bilinear forms in the Witt-Grothendieck ring of  $K$ , deriving their basic properties as well as computing several of their classical invariants such as determinant, signature with respect to an ordering and Hasse invariant. Their relation to exterior powers was also given. For a given symmetric bilinear form  $\varphi$  over  $K$ , there was a distinction drawn between factorial symmetric powers  $S^k\varphi$  and non-factorial symmetric powers  $S^k\varphi$ . Factorial symmetric powers were seen to have the advantages of being computable for any given symmetric bilinear form and of being independent of basis. However, effectively they are useful only in characteristic 0. On the other hand, non-factorial symmetric powers are independent of characteristic but require the form to be diagonalised, i.e., a basis needs to be chosen. In this paper, we deal with non-factorial symmetric powers of forms, hereafter referred to simply as symmetric powers. Since  $\text{char}(K) \neq 2$ , we use the one-to-one correspondence between symmetric bilinear forms and quadratic forms.

Notation and terminology is borrowed from Lam [8] and Scharlau [10]. A diagonalised quadratic form over  $K$  with coefficients  $a_1, \dots, a_m \in K^\times$  is denoted by  $\langle a_1, \dots, a_m \rangle$ . The hyperbolic plane  $\langle 1, -1 \rangle$  is denoted by  $\mathbb{H}$ . If  $\varphi$  and  $\psi$  are forms over  $K$ , then  $\varphi \simeq \psi$  means that these forms are isometric. The tensor product of the 1-dimensional form  $\langle n \rangle$  with  $\varphi$  is denoted  $\langle n \rangle\varphi$  to be distinguished from  $n$  copies of  $\varphi$ , which is written as  $n \times \varphi$ .

## 2. Preliminaries

**2.1. Exterior powers.** Bourbaki defined the concept of exterior power of a symmetric bilinear form in [1, IX, Section 1, (37)]:

**Definition 2.1.** Let  $V$  be a vector space of dimension  $m$  over  $K$ . Let  $\varphi : V \times V \rightarrow K$  be a non-singular bilinear form and let  $k$  be a positive integer,  $k \leq m$ . We define the *k-fold exterior power* of  $\varphi$ ,

$$\Lambda^k \varphi : \Lambda^k V \times \Lambda^k V \rightarrow K,$$

by

$$\Lambda^k \varphi(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) = \det(\varphi(x_i, y_j))_{1 \leq j \leq k}.$$

The bilinear extension of this form defines  $\Lambda^k \varphi$  everywhere on  $\Lambda^k V \times \Lambda^k V$ . We define  $\Lambda^0 \varphi := \langle 1 \rangle$ . Also, for  $k > m$ , we define  $\Lambda^k \varphi := 0$ , the zero form.

We use the following two results on exterior powers of hyperbolic forms:

**Proposition 2.2** ([3, Proposition 5.6]). *Let  $\phi \simeq h \times \mathbb{H}$ , where  $h \in \mathbb{N}$  and  $k$  odd with  $1 \leq k \leq 2h - 1$ . Then*

$$\Lambda^k \phi \simeq \frac{1}{2} \binom{2h}{k} \times \mathbb{H}.$$

**Proposition 2.3** ([3, Proposition 5.7]). *Let  $\phi \simeq h \times \mathbb{H}$ , where  $h \in \mathbb{N}$ ,  $k = 2\ell$  and  $0 \leq \ell \leq h$ . Then*

$$\Lambda^k \phi = \Lambda^{2\ell} \phi \simeq \binom{h}{\ell} \times \langle (-1)^\ell \rangle \perp \frac{1}{2} \left( \binom{2h}{2\ell} - \binom{h}{\ell} \right) \times \mathbb{H}.$$

## 2.2. Symmetric powers

**Definition 2.4** ([9, Adapted statement of Proposition 3.5]). Let  $V$  be a vector space of dimension  $m$  over  $K$ . Let  $\varphi$  be a symmetric bilinear form over  $K$  with  $\varphi \simeq \langle a_1, \dots, a_m \rangle$ . Let  $k$  be a non-negative integer. We define the  $k$ -fold symmetric power of  $\varphi$ ,

$$S^k \varphi : S^k V \times S^k V \rightarrow K,$$

by

$$S^k \varphi = \bigsqcup_{\substack{1 \leq i_1 < \dots < i_\ell \leq m \\ k_{i_1} + \dots + k_{i_\ell} = k}} \langle a_{i_1}^{k_{i_1}} \dots a_{i_\ell}^{k_{i_\ell}} \rangle.$$

**Remark 2.5.** The form  $S^k\varphi$  has dimension  $\binom{m+k-1}{k}$ .

**Remark 2.6.** Clearly,  $S^0\varphi = \langle 1 \rangle$  and  $S^1\varphi = \varphi$ .

**Remark 2.7.** We note that  $S^k$  preserves isometries, i.e.,  $\varphi \simeq \psi \Rightarrow S^k\varphi \simeq S^k\psi$ . In general, the converse is not true; consider a field where  $\langle 1 \rangle \neq \langle -1 \rangle$ . Let  $\varphi = \langle 1, 1 \rangle \perp \mathbb{H}$  and  $\psi = \langle -1, -1 \rangle \perp \mathbb{H}$ . Then  $\varphi \neq \psi$  but  $S^2\varphi \simeq S^2\psi = 4 \times \langle 1 \rangle \perp 3 \times \mathbb{H}$ .

**Remark 2.8.** The  $k$ -fold exterior power is always a subform of the  $k$ -fold symmetric power. Therefore,  $\Lambda^k\varphi$  isotropic  $\Rightarrow S^k\varphi$  isotropic.

**Lemma 2.9.**  $S^k(m \times \langle 1 \rangle) = \binom{m+k-1}{k} \times \langle 1 \rangle$  and  $S^k(m \times \langle -1 \rangle) = \binom{m+k-1}{k} \times \langle (-1)^k \rangle$ .

The next two propositions give properties that are very useful for the computations in Sections 3 and 4.

**Proposition 2.10** ([7, p. 47]). *Let  $\varphi$  and  $\psi$  be symmetric bilinear forms over  $K$  and let  $k$  be a positive integer. Then*

$$S^k(\varphi \perp \psi) = \bigoplus_{i+j=k} S^i\varphi \otimes S^j\psi.$$

**Proposition 2.11** ([9, Proposition 4.4]). *Let  $\varphi$  be an  $n$ -dimensional symmetric bilinear form over  $K$  and let  $k$  be a positive integer. Then*

$$S^k\varphi = \bigoplus_{i=0}^{\lceil k/2 \rceil} \binom{n+i-1}{i} \times \Lambda^{k-2i}\varphi.$$

### 3. Symmetric Powers of Hyperbolic Forms

For each dimension of even parity, there exists exactly one hyperbolic form up to isometry. In the other words, if  $\varphi$  is a hyperbolic form of dimension  $2h$ , then  $\varphi \simeq h \times \mathbb{H}$ . In this section, we compute the symmetric

powers of these forms. For computational convenience, we often do not state the specific number of hyperbolic planes. Instead, we shall use the word ‘‘Hyp’’ to represent the number of hyperbolic planes required in order to give full dimension to the particular symmetric power in question.

**Proposition 3.1.** *Let  $k$  be any positive odd integer,  $h \in \mathbb{N}$ . Then*

$$S^k(h \times \mathbb{H}) = \frac{1}{2} \binom{2h+k-1}{k} \times \mathbb{H}.$$

**Proof.**

$$\begin{aligned} S^k(h \times \mathbb{H}) &= S^k(h \times \langle 1 \rangle \perp h \times \langle -1 \rangle) \\ &= \bigsqcup_{i+j=k} S^i(h \times \langle 1 \rangle) \otimes S^j(h \times \langle -1 \rangle) \\ &= \bigsqcup_{\substack{i+j=k \\ j \text{ even}}} S^i(h \times \langle 1 \rangle) \otimes S^j(h \times \langle -1 \rangle) \perp \bigsqcup_{\substack{i+j=k \\ j \text{ odd}}} S^i(h \times \langle 1 \rangle) \\ &\quad \otimes S^j(h \times \langle -1 \rangle) \quad [\text{by Proposition 2.10}] \\ &= \bigsqcup_{\substack{i+j=k \\ j \text{ even}}} \left( \binom{h+i-1}{i} \times \langle 1 \rangle \right) \otimes \left( \binom{h+j-1}{j} \times \langle 1 \rangle \right) \\ &\quad \perp \bigsqcup_{\substack{i+j=k \\ j \text{ odd}}} \left( \binom{h+i-1}{i} \times \langle 1 \rangle \right) \otimes \left( \binom{h+j-1}{j} \times \langle -1 \rangle \right) \quad [\text{by Lemma 2.9}] \\ &= \sum_{\substack{i+j=k \\ j \text{ even}}} \binom{h+i-1}{i} \binom{h+j-1}{j} \times \langle 1 \rangle \\ &\quad \perp \sum_{\substack{i+j=k \\ j \text{ odd}}} \binom{h+i-1}{i} \binom{h+j-1}{j} \times \langle -1 \rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{i+j=k \\ j \text{ even}}} \binom{h+i-1}{i} \binom{h+j-1}{j} \times \langle 1 \rangle \\
&\quad \perp \sum_{\substack{i+j=k \\ j \text{ even}}} \binom{h+j-1}{j} \binom{h+i-1}{i} \times \langle -1 \rangle.
\end{aligned}$$

From the expression above, we observe that there is the same number of summands  $\langle 1 \rangle$  as summands  $\langle -1 \rangle$ . Given that the dimension of  $S^k(h \times \mathbb{H})$  is  $\binom{2h+k-1}{k}$ , we conclude that

$$S^k(h \times \mathbb{H}) = \frac{1}{2} \binom{2h+k-1}{k} \times \mathbb{H}.$$

□

We now seek to compute  $S^k(h \times \mathbb{H})$  for  $k$  an arbitrary even natural number. First, we need two lemmas:

**Lemma 3.2.**

$$\sum_{j=0}^r (-1)^j \binom{2p+j-1}{j} \binom{p}{r-j} = (-1)^r \binom{p+r-1}{r}.$$

**Proof.** By the Vandermonde convolution (see [4], for example), for any integer  $p$  and any positive integer  $r$ ,

$$\sum_{j=0}^r \binom{-2p}{j} \binom{p}{r-j} = \binom{-p}{r}.$$

By applying the “minus transformation”  $\binom{-q}{i} = (-1)^i \binom{q+i-1}{i}$  to both sides, we get the required identity. □

**Lemma 3.3** [Pascal's rule]. *For any  $r \in \mathbb{N}$ ,  $1 \leq s \leq r$ , we have that*

$$\binom{r-1}{s} + \binom{r-1}{s-1} = \binom{r}{s}.$$

**Remark 3.4.** In several of the remaining proofs, we use the following property of binomial coefficients, which is easy to derive:

$$\binom{r}{s} = \frac{r}{s} \binom{r-1}{s-1}.$$

**Proposition 3.5.** *Let  $k$  be any non-negative even integer,  $k = 2\ell$ ,  $h \in \mathbb{N}$ . Then*

$$S^k(h \times \mathbb{H}) = S^{2\ell}(h \times \mathbb{H}) = \binom{h+\ell-1}{\ell} \times \langle 1 \rangle \perp \frac{1}{2} \left( \binom{2h+2\ell-1}{2\ell} - \binom{h+\ell-1}{\ell} \right) \times \mathbb{H}.$$

**Proof.** Using the formula

$$S^k \phi = \prod_{i=0}^{\lceil k/2 \rceil} \binom{n+i-1}{i} \times \Lambda^{k-2i} \phi, \quad \text{[by Proposition 2.11]}$$

we get

$$\begin{aligned} S^k(h \times \mathbb{H}) &= S^{2\ell}(h \times \mathbb{H}) \\ &= \prod_{i=0}^{\ell} \binom{2h+i-1}{i} \times \Lambda^{2\ell-2i}(h \times \mathbb{H}) \\ &= \prod_{i=0}^{\ell} \binom{2h+i-1}{i} \\ &\quad \times \left( \binom{h}{\ell-i} \times \langle (-1)^{\ell-i} \rangle \perp \frac{1}{2} \left( \binom{2h}{2(\ell-i)} - \binom{h}{\ell-i} \right) \times \mathbb{H} \right) \end{aligned}$$

[by Proposition 2.3]

$$\begin{aligned}
&= \prod_{i=0}^{\ell} \binom{2h+i-1}{i} \binom{h}{\ell-i} \times \langle (-1)^{\ell-i} \rangle \perp \text{Hyp} \\
&= \prod_{i=0}^{\ell} \binom{2h+i-1}{i} \binom{h}{\ell-i} \times \langle (-1)^{i-\ell} \rangle \perp \text{Hyp} \\
&= \sum_{i=0}^{\ell} (-1)^i \binom{2h+i-1}{i} \binom{h}{\ell-i} \times \langle (-1)^{-\ell} \rangle \perp \text{Hyp} \\
&= (-1)^{\ell} \binom{h+\ell-1}{\ell} \times \langle (-1)^{-\ell} \rangle \perp \text{Hyp} \quad [\text{by Lemma 3.2}] \\
&= \binom{h+\ell-1}{\ell} \times \langle 1 \rangle \perp \text{Hyp}.
\end{aligned}$$

□

#### 4. Symmetric Powers of Trace Forms on Symbol Algebras

In [5], Garibaldi et al. used the exterior powers of trace forms to define certain cohomological invariants of central simple algebras. Here, we compute the symmetric powers of trace forms of symbol algebras, which are considered to be the “building blocks” of central simple algebras.

Let  $n$  be an arbitrary positive integer and let  $K$  contain a primitive  $n$ -th root of unity  $\omega$ . Let  $a, b \in K^\times$  and let  $S$  be the algebra over  $K$  generated by elements  $x$  and  $y$ , where

$$x^n = a, y^n = b \quad \text{and} \quad yx = \omega xy.$$

We call this algebra a *symbol algebra* (see [6, Chapter 1, Section 2]) and denote it as  $(a, b; n, K, \omega)$ . In [2, Section 11], Draxl shows it to be a central simple algebra over  $K$  of degree  $n$ .



Let  $A$  be a central simple algebra of degree  $n$  over a field  $K$  of characteristic different from 2. We write  $T_A : A \rightarrow K$  for the quadratic trace form

$$T_A(z) = \text{Trd}_A(z^2) \quad \text{for } z \in A,$$

where  $\text{Trd}_A$  is the reduced trace of  $A$ .

In [3], we proved the following:

**Proposition 4.1** ([3, Propositions 2.1 and 3.1]). *For  $S = (a, b; n, K, \omega)$ , we have*

$$(i) \quad T_s \simeq \langle (-1)^{n/2} \rangle \perp \frac{n^2 - 1}{2} \times \mathbb{H}, \quad \text{for } n \text{ odd.}$$

$$(ii) \quad T_s \simeq \langle n \rangle \langle 1, a, b, (-1)^{n/2} ab \rangle \perp \frac{n^2 - 4}{2} \times \mathbb{H}, \quad \text{for } n \text{ even.}$$

**Proposition 4.2.** *Let  $n$  be odd and  $k \geq 0$ . Then*

$$S^k T_S = \begin{cases} \left( \frac{\frac{n^2+k-2}{2}}{\frac{k-1}{2}} \right) \times \langle (-1)^{\frac{n-1}{2}} \rangle \perp \text{Hyp}, & \text{if } k \text{ is odd;} \\ \left( \frac{n^2+k-1}{\frac{k}{2}} \right) \times \langle 1 \rangle \perp \text{Hyp}, & \text{if } k \text{ is even.} \end{cases}$$

**Proof.** Let  $k$  be odd. Then

$$\begin{aligned} S^k T_S &= S^k \left( \langle (-1)^{\frac{n-1}{2}} \rangle \perp \frac{n^2 - 1}{2} \times \mathbb{H} \right) && \text{[by Proposition 4.1]} \\ &= \bigsqcup_{i+j=k} S^i \left( \langle (-1)^{\frac{n-1}{2}} \rangle \right) \otimes S^j \left( \frac{n^2 - 1}{2} \times \mathbb{H} \right) && \text{[by Proposition 2.10]} \\ &= \bigsqcup_{j \text{ even}} S^i \left( \langle (-1)^{\frac{n-1}{2}} \rangle \right) \otimes S^j \left( \frac{n^2 - 1}{2} \times \mathbb{H} \right) \\ &\quad \perp \bigsqcup_{j \text{ odd}} S^i \left( \langle (-1)^{\frac{n-1}{2}} \rangle \right) \otimes S^j \left( \frac{n^2 - 1}{2} \times \mathbb{H} \right) \end{aligned}$$

$$\begin{aligned}
&= \bigsqcup_{j \text{ even}} S^i \left( \langle (-1)^{\frac{n-1}{2}} \rangle \right) \otimes S^j \left( \frac{n^2-1}{2} \times \mathbb{H} \right) \perp \text{Hyp} \\
&\hspace{20em} \text{[by Proposition 3.1]} \\
&= \bigsqcup_{j \text{ even}} \left( \langle (-1)^{\frac{(n-1)i}{2}} \rangle \otimes S^j \left( \frac{n^2-1}{2} \times \mathbb{H} \right) \right) \perp \text{Hyp} \quad \text{[by Lemma 2.9]} \\
&= \bigsqcup_{j \text{ even}} \left( \langle (-1)^{\frac{(n-1)}{2}} \rangle \otimes \left( \binom{\frac{n^2-1+j-2}{2}}{\frac{j}{2}} \times \langle 1 \rangle \right. \right. \\
&\quad \left. \left. \perp \left( \binom{n^2-1+j-2}{j} - \binom{n^2-1+j-2}{\frac{j}{2}} \right) \times \mathbb{H} \right) \right) \perp \text{Hyp} \quad \text{[by Proposition 3.5]} \\
&= \sum_{j \text{ even}} \left( \binom{\frac{n^2+j-3}{2}}{\frac{j}{2}} \times \langle (-1)^{\frac{n-1}{2}} \rangle \right) \perp \text{Hyp} \\
&= \sum_{j'=0}^{\frac{k-1}{2}} \binom{\frac{n^2-1}{2} + j' - 1}{j'} \times \langle (-1)^{\frac{n-1}{2}} \rangle \perp \text{Hyp} \\
&\hspace{20em} \text{[changing the index, } j' = j/2 \text{]} \\
&= \binom{\frac{n^2+k-2}{2}}{\frac{k-1}{2}} \times \langle (-1)^{\frac{n-1}{2}} \rangle \perp \text{Hyp.} \quad \text{[by repeated use of Lemma 3.3]}
\end{aligned}$$

Now let  $k$  be even. Then

$$\begin{aligned}
S^k T_S &= S^k \left( \langle (-1)^{\frac{n-1}{2}} \rangle \perp \frac{n^2-1}{2} \times \mathbb{H} \right) \hspace{10em} \text{[by Proposition 4.1]} \\
&= \bigsqcup_{i+j=k} S^i \left( \langle (-1)^{\frac{n-1}{2}} \rangle \right) \otimes S^j \left( \frac{n^2-1}{2} \times \mathbb{H} \right) \hspace{2em} \text{[by Proposition 2.10]}
\end{aligned}$$

$$\begin{aligned}
 &= \bigsqcup_{j \text{ even}} S^i \left( \langle (-1)^{\frac{n-1}{2}} \rangle \right) \otimes S^j \left( \frac{n^2-1}{2} \times \mathbb{H} \right) \\
 &\quad \perp \bigsqcup_{j \text{ odd}} S^i \left( \langle (-1)^{\frac{n-1}{2}} \rangle \right) \otimes S^j \left( \frac{n^2-1}{2} \times \mathbb{H} \right) \\
 &= \bigsqcup_{j \text{ even}} S^i \left( \langle (-1)^{\frac{n-1}{2}} \rangle \right) \otimes S^j \left( \frac{n^2-1}{2} \times \mathbb{H} \right) \perp \text{Hyp} \\
 &\hspace{25em} \text{[by Proposition 3.1]} \\
 &= \bigsqcup_{j \text{ even}} \left( \langle (-1)^{\frac{(n-1)j}{2}} \rangle \otimes S^j \left( \frac{n^2-1}{2} \times \mathbb{H} \right) \right) \perp \text{Hyp} \quad \text{[by Lemma 2.9]} \\
 &= \bigsqcup_{j \text{ even}} \left( \langle 1 \rangle \otimes \left( \left( \frac{n^2-1+j-2}{2} \right) \times \langle 1 \rangle \right. \right. \\
 &\quad \left. \left. \perp \left( \left( \frac{n^2-1+j-2}{j} \right) - \left( \frac{n^2-1+j-2}{\frac{j}{2}} \right) \right) \times \mathbb{H} \right) \right) \text{[by Proposition 3.5]} \\
 &= \sum_{j \text{ even}} \left( \frac{n^2+j-3}{\frac{j}{2}} \right) \times \langle 1 \rangle \perp \text{Hyp} \\
 &= \sum_{j'=0}^{\frac{k}{2}} \left( \frac{n^2-1}{2} + j' - 1 \right) \times \langle 1 \rangle \perp \text{Hyp} \\
 &\hspace{25em} \text{[changing the index, } j' = j/2 \text{]} \\
 &= \left( \frac{n^2+k-1}{\frac{k}{2}} \right) \times \langle 1 \rangle \perp \text{Hyp.} \quad \text{[by repeated use of Lemma 3.3]}
 \end{aligned}$$

□

**Proposition 4.3.** *Let  $n$  be even and  $k$  be odd. We write  $T_S \simeq q_S \perp m \times \mathbb{H}$ ,*

*where  $q_S \simeq \langle n \rangle \langle 1, a, b(-1)^{\frac{n}{2}} ab \rangle$  and  $m = \frac{n^2 - 4}{2}$ . Then for  $k > 1$ ,*

$$S^k T_S = \begin{cases} \frac{2m + 2k + 4}{k - 1} \binom{(2m+k+3)/2}{(k-3)/2} \times q_S \perp \text{Hyp}, & \text{if } n \equiv 0 \pmod{4}; \\ \frac{2m + 6}{k - 1} \binom{(2m+k+3)/2}{(k-3)/2} \times q_S \perp \text{Hyp}, & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

**Proof.**

$$\begin{aligned} S^k T_S &= S^k(q_S \perp m \times \mathbb{H}) \\ &= \bigsqcup_{i+j=k} S^i q_S \otimes S^j(m \times \mathbb{H}) && \text{[by Proposition 2.10]} \\ &= \bigsqcup_{j \text{ even}} S^i q_S \otimes S^j(m \times \mathbb{H}) \perp \bigsqcup_{j \text{ odd}} S^i q_S \otimes S^j(m \times \mathbb{H}) \\ &= \bigsqcup_{j \text{ even}} S^i q_S \otimes S^j(m \times \mathbb{H}) \perp \text{Hyp} && \text{[by Proposition 3.1]} \\ &= \bigsqcup_{j \text{ even}} S^i q_S \otimes \left( \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \times \langle 1 \rangle \right. \\ &\quad \left. \perp \frac{1}{2} \left( \binom{2m + j - 1}{j} - \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \right) \times \mathbb{H} \right) \perp \text{Hyp} && \text{[by Proposition 3.5]} \\ &= \bigsqcup_{j \text{ even}} S^i q_S \otimes \left( \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \times \langle 1 \rangle \right) \perp \text{Hyp} \\ &= \sum_{j \text{ even}} \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \times S^i q_S \perp \text{Hyp} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j \text{ even}} \left( m + \frac{j}{2} - 1 \right) \times \left( \prod_{\ell=0}^{(i+1)/2} \binom{3+\ell}{\ell} \times \Lambda^{i-2\ell} q_S \right) \perp \text{Hyp} \\
 &\hspace{20em} [\text{by Proposition 2.11}] \\
 &= \sum_{j \text{ even}} \left( m + \frac{j}{2} - 1 \right) \times \left( \binom{3 + \frac{i-1}{2}}{\frac{i-1}{2}} \times \Lambda^1 q_S \right) \\
 &\quad \perp \left( \binom{3 + \frac{i-3}{2}}{\frac{i-3}{2}} \times \Lambda^3 q_S \right) \perp \text{Hyp} \text{ [since for larger } k, \Lambda^k \text{ is zero]} \\
 &= \sum_{j \text{ even}} \left( m + \frac{j}{2} - 1 \right) \times \left( \binom{3 + \frac{i-1}{2}}{\frac{i-1}{2}} \times q_S \right) \\
 &\quad \perp \left( \binom{3 + \frac{i-3}{2}}{\frac{i-3}{2}} \times \langle (-1)^{n/2} \rangle \otimes q_S \right) \perp \text{Hyp} \\
 &= \sum_{j \text{ even}} \left( m + \frac{j}{2} - 1 \right) \binom{3 + \frac{i-1}{2}}{\frac{i-1}{2}} \times q_S \\
 &\quad \perp \sum_{j \text{ even}} \left( m + \frac{j}{2} - 1 \right) \binom{3 + \frac{i-3}{2}}{\frac{i-3}{2}} \times \langle (-1)^{n/2} \rangle \otimes q_S \perp \text{Hyp} \\
 &= \sum_{j \text{ even}} \left( m + \frac{j}{2} - 1 \right) \binom{3 + \frac{k-1}{2} - \frac{j}{2}}{\frac{k-1}{2} - \frac{j}{2}} \times q_S \\
 &\quad \perp \sum_{j \text{ even}} \left( m + \frac{j}{2} - 1 \right) \binom{3 + \frac{k-3}{2} - \frac{j}{2}}{\frac{k-3}{2} - \frac{j}{2}} \times \langle (-1)^{n/2} \rangle \otimes q_S \perp \text{Hyp}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{j'=0}^{\frac{k-1}{2}} \binom{m+j'-1}{j'} \binom{3+\frac{k-1}{2}-j'}{\frac{k-1}{2}-j'} \times q_S \\
&\quad \perp \sum_{j' \text{ even}} \binom{m+j'-1}{j'} \binom{3+\frac{k-3}{2}-j'}{\frac{k-3}{2}-j'} \times \langle (-1)^{n/2} \rangle \otimes q_S \perp \text{Hyp} \\
&\hspace{15em} [\text{changing the index, } j' = j/2] \\
&= \binom{m+3+\frac{k-1}{2}}{\frac{k-1}{2}} \times q_S \perp \binom{m+3+\frac{k-3}{2}}{\frac{k-3}{2}} \times \langle (-1)^{n/2} \rangle \otimes q_S \perp \text{Hyp}. \\
&\hspace{15em} [\text{by [4, Equation (3)}]
\end{aligned}$$

Now if  $n \equiv 0 \pmod{4}$

$$\begin{aligned}
S^k T_S &= \left( \binom{m+3+\frac{k-1}{2}}{\frac{k-1}{2}} + \binom{m+3+\frac{k-3}{2}}{\frac{k-3}{2}} \right) \times q_S \perp \text{Hyp} \\
&= \frac{2m+2k+4}{k-1} \binom{2m+k+3}{\frac{k-3}{2}} \times q_S \perp \text{Hyp}. \quad [\text{by Remark 3.4}]
\end{aligned}$$

On the other hand, if  $n \equiv 2 \pmod{4}$

$$\begin{aligned}
S^k T_S &= \left( \binom{m+3+\frac{k-1}{2}}{\frac{k-1}{2}} - \binom{m+3+\frac{k-3}{2}}{\frac{k-3}{2}} \right) \times q_S \perp \text{Hyp} \\
&= \frac{2m+6}{k-1} \binom{2m+k+3}{\frac{k-3}{2}} \times q_S \perp \text{Hyp}. \quad [\text{by Remark 3.4}]
\end{aligned}$$

□

**Proposition 4.4.** *Let  $n$  and  $k$  be even. We write  $T_S \simeq q_S \perp m \times \mathbb{H}$ ,*

*where  $q_S \simeq \langle n \rangle \langle 1, a, b, (-1)^{\frac{n}{2}} ab \rangle$  and  $m = \frac{n^2 - 4}{2}$ . Then for  $k > 1$ ,*

$$S^k T_S = \begin{cases} \left( 1 + \frac{(m+3+k/2)(m+2+k/2)}{(k/2)(k/2-1)} \right) \binom{m+1+k/2}{k/2-2} \times \langle 1 \rangle \perp \text{Hyp}, & \text{if } n \equiv 0 \pmod{4}; \\ \left( \frac{(m+3+k/2)(m+2+k/2)}{(k/2)(k/2-1)} - 1 \right) \binom{m+1+k/2}{k/2-2} \times \langle 1 \rangle \perp \text{Hyp}, & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

**Proof.**

$$\begin{aligned} S^k T_S &= S^k(q_S \perp m \times \mathbb{H}) \\ &= \bigsqcup_{i+j=k} S^i q_S \otimes S^j(m \times \mathbb{H}) && \text{[by Proposition 2.10]} \\ &= \bigsqcup_{j \text{ even}} S^i q_S \otimes S^j(m \times \mathbb{H}) \perp \bigsqcup_{j \text{ odd}} S^i q_S \otimes S^j(m \times \mathbb{H}) \\ &= \bigsqcup_{j \text{ even}} S^i q_S \otimes S^j(m \times \mathbb{H}) \perp \text{Hyp} && \text{[by Proposition 3.1]} \\ &= \bigsqcup_{j \text{ even}} S^i q_S \otimes \left( \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \times \langle 1 \rangle \right) \\ &\quad \perp \frac{1}{2} \left( \binom{2m + j - 1}{j} - \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \right) \times \mathbb{H} \perp \text{Hyp} && \text{[by Proposition 3.5]} \\ &= \bigsqcup_{j \text{ even}} S^i q_S \otimes \left( \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \times \langle 1 \rangle \right) \perp \text{Hyp} \\ &= \sum_{j \text{ even}} \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \times S^i q_S \perp \text{Hyp} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j \text{ even}} \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \times \left( \binom{i/2}{\ell=0} \binom{3 + \ell}{\ell} \times \Lambda^{i-2\ell} q_S \right) \perp \text{Hyp} \\
&\hspace{20em} [\text{by Proposition 2.11}] \\
&= \sum_{j \text{ even}} \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \times \left( \binom{3 + \frac{i}{2}}{\frac{i}{2}} \times \Lambda^0 q_S \perp \binom{3 + \frac{i-2}{2}}{\frac{i-2}{2}} \times \Lambda^2 q_S \right. \\
&\quad \left. \perp \binom{3 + \frac{i-4}{2}}{\frac{i-4}{2}} \times \Lambda^4 q_S \right) \perp \text{Hyp} \quad [\text{since for larger } k, \Lambda^k \text{ is zero}] \\
&= \sum_{j \text{ even}} \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \times \left( \binom{3 + \frac{i}{2}}{\frac{i}{2}} \times \langle 1 \rangle \perp \binom{3 + \frac{i-2}{2}}{\frac{i-2}{2}} \times \langle a, b, ab \rangle \right. \\
&\quad \left. \otimes \langle 1, (-1)^{n/2} \rangle \perp \binom{3 + \frac{i-4}{2}}{\frac{i-4}{2}} \times \langle (-1)^{n/2} \rangle \right) \perp \text{Hyp} \\
&= \sum_{j \text{ even}} \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \binom{3 + \frac{k-j}{2}}{\frac{k-j}{2}} \times \langle 1 \rangle \\
&\quad \perp \sum_{j \text{ even}} \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \binom{3 + \frac{k-j-2}{2}}{\frac{k-j-2}{2}} \times \langle a, b, ab \rangle \otimes \langle 1, (-1)^{n/2} \rangle \\
&\quad \perp \sum_{j \text{ even}} \binom{m + \frac{j}{2} - 1}{\frac{j}{2}} \binom{3 + \frac{k-j-4}{2}}{\frac{k-j-4}{2}} \times \langle (-1)^{n/2} \rangle \perp \text{Hyp}
\end{aligned}$$



$$\begin{aligned}
 &= \sum_{j'=0}^{k/2} \binom{m+j'-1}{j'} \binom{3+k/2-j'}{k/2-j'} \times \langle 1 \rangle \\
 &\perp \sum_{j'=0}^{k/2-1} \binom{m+j'-1}{j'} \binom{3+k/2-1-j'}{k/2-1-j'} \times \langle a, b, ab \rangle \otimes \langle 1, (-1)^{n/2} \rangle \\
 &\perp \sum_{j'=0}^{k/2-2} \binom{m+j'-1}{j'} \binom{3+k/2-2-j'}{k/2-2-j'} \times \langle (-1)^{n/2} \rangle \perp \text{Hyp} \\
 &\hspace{15em} [\text{changing the index, } j' = j/2] \\
 &= \binom{m+3+k/2}{k/2} \times \langle 1 \rangle \perp \binom{m+2+k/2}{k/2-1} \times \langle a, b, ab \rangle \otimes \langle 1, (-1)^{n/2} \rangle \\
 &\perp \binom{m+1+k/2}{k/2-2} \times \langle (-1)^{n/2} \rangle \perp \text{Hyp.} \quad [\text{by [4, Equation (3)]]
 \end{aligned}$$

We now examine separately the cases of  $n \equiv 0 \pmod{4}$  and  $n \equiv 2 \pmod{4}$ .

Suppose  $n \equiv 0 \pmod{4}$ . In this case, we have  $(-1)^{n/2} = 1$  and so

$$\begin{aligned}
 S^k T_S &= \left( \binom{m+3+k/2}{k/2} + \binom{m+1+k/2}{k/2-2} \right) \times \langle 1 \rangle \perp 2 \binom{m+2+k/2}{k/2-1} \times \langle a, b, ab \rangle \perp \text{Hyp} \\
 &= \left( 1 + \frac{(m+3+k/2)(m+2+k/2)}{(k/2)(k/2-1)} \right) \binom{m+1+k/2}{k/2-2} \times \langle 1 \rangle \\
 &\hspace{15em} \perp 2 \binom{m+2+k/2}{k/2-1} \times \langle a, b, ab \rangle \perp \text{Hyp} \\
 &= \left( 1 + \frac{(m+3+k/2)(m+2+k/2)}{(k/2)(k/2-1)} \right) \binom{m+1+k/2}{k/2-2} \times \langle 1 \rangle \perp \text{Hyp}.
 \end{aligned}$$

Now suppose  $n \equiv 2 \pmod{4}$ . In this case, we have  $(-1)^{n/2} = -1$  and so

$$\begin{aligned} S^k T_S &= \binom{m+3+k/2}{k/2} \times \langle 1 \rangle \perp \binom{m+1+k/2}{k/2-2} \times \langle -1 \rangle \perp \text{Hyp} \\ &= \left( \binom{m+3+k/2}{k/2} - \binom{m+1+k/2}{k/2-2} \right) \times \langle 1 \rangle \perp \text{Hyp} \\ &= \left( \frac{(m+3+k/2)(m+2+k/2)}{(k/2)(k/2-1)} \right) \binom{m+1+k/2}{k/2-2} \times \langle 1 \rangle \perp \text{Hyp}. \end{aligned}$$

□

### 5. Summary of Results

In Table 1, we give a summary of the above results on symmetric powers alongside the corresponding results on exterior powers computed in [3, Section 5].

Let  $\varphi \simeq h \times \mathbb{H}$  for arbitrary  $h \in \mathbb{N}$ . Then

**Table 1.** Exterior and symmetric powers of hyperbolic forms over  $K$

$k$	$\Lambda^k \varphi$	$S^k \varphi$
odd	Hyp	Hyp
even	$\binom{h}{\frac{k}{2}} \times \langle (-1)^{\frac{k}{2}} \rangle \perp \text{Hyp}$	$\binom{h + \frac{k}{2} - 1}{\frac{k}{2}} \times \langle 1 \rangle \perp \text{Hyp}$

Let  $S = (a, b; n, K, \omega)$  be a symbol algebra of degree  $n$  as described in Section 4. Let  $\varphi = T_S = \langle n \rangle \langle 1, a, b, (-1)^{\frac{n}{2}} ab \rangle$  and  $m = \frac{n^2 - 4}{2}$ . In Table 2, we summarise the results for exterior and symmetric powers of  $T_S$ .

**Table 2.** Exterior and symmetric powers of  $T_S$

$n$	$k$	$\Lambda^k \varphi$	$S^k \varphi$
odd	odd	$\binom{\frac{n^2-1}{2}}{\frac{k-1}{2}} \times \langle (-1)^{\frac{k-1}{2}} \rangle \perp \text{Hyp}$	$\binom{\frac{n^2+k-2}{2}}{\frac{k-1}{2}} \times \langle (-1)^{\frac{n-1}{2}} \rangle \perp \text{Hyp}$
odd	even	$\binom{\frac{n^2-1}{2}}{\frac{k}{2}} \times \langle (-1)^{\frac{k}{2}} \rangle \perp \text{Hyp}$	$\binom{\frac{n^2+k-1}{2}}{\frac{k}{2}} \times \langle 1 \rangle \perp \text{Hyp}$
$0 \pmod{4}$	odd	$\binom{m+1}{\frac{k-1}{2}} \times \langle (-1)^{\frac{n(k-1)}{4}} \rangle q_S \perp \text{Hyp}$	$\frac{2m+2k+4}{k-1} \binom{\frac{2m+k+3}{2}}{\frac{k-3}{2}} \times q_S \perp \text{Hyp}$
$0 \pmod{4}$	even	$\binom{\frac{n^2}{2}}{\frac{k}{2}} \times \langle 1 \rangle \perp \text{Hyp}$	$\left(1 + \frac{(m+3+k/2)(m+2+k/2)}{(k/2)(k/2-1)}\right) \binom{\frac{2m+2+k}{2}}{\frac{k-4}{2}} \times \langle 1 \rangle \perp \text{Hyp}$
$2 \pmod{4}$	odd	$\binom{m+1}{\frac{k-1}{2}} \times \langle (-1)^{\frac{n(k-1)}{4}} \rangle q_S \perp \text{Hyp}$	$\frac{2m+6}{k-1} \binom{\frac{2m+k+3}{2}}{\frac{k-3}{2}} \times q_S \perp \text{Hyp}$
$2 \pmod{4}$	even	$\begin{cases} \left(1 - \frac{2k}{n^2}\right) \binom{\frac{n^2}{2}}{\frac{k}{2}} \times \langle (-1)^{\frac{k}{2}} \rangle \perp \text{Hyp}, k \leq \frac{n^2}{2} \\ \left(\frac{2k}{n^2} - 1\right) \binom{\frac{n^2}{2}}{\frac{k}{2}} \times \langle (-1)^{\frac{k+2}{2}} \rangle \perp \text{Hyp}, k > \frac{n^2}{2} \end{cases}$	$\left(\frac{(m+3+k/2)(m+2+k/2)}{(k/2)(k/2-1)} - 1\right) \binom{\frac{2m+2+k}{2}}{\frac{k-4}{2}} \times \langle 1 \rangle \perp \text{Hyp}$

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