CROSSING BOUNDARIES INTO SECONDARY MATHEMATICS CONCEPTS: THE CASE OF FUNCTIONS

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This paper reports on the use of a particular function machine which was based on the context of the 'faulty oven' to develop conceptual understanding of the concept of function among 4th class children. While the concept of functions is traditionally associated with secondary level mathematics, this research shows that the identification of appropriate models and contexts supports primary level children in accessing complex mathematical ideas. This approach was researched, developed, and tested in classrooms as part of a Lesson Study project with pre-service elementary teachers and mathematics teacher educators.

INTRODUCTION

Agreement exists that "The quality of teaching of mathematics … is essential to improving mathematics proficiency" (EGFSN, 2008, p. 7). Research highlights the significance of mathematics at primary level. It is reported that early successes in mathematics facilitate the development of the necessary foundation, interest and motivation to successfully study mathematics during formal education (NCCA, 2006; EGFSN, 2008). Therefore while there was a time when people doubted that kindergarten and elementary grade children were capable of learning algebra, over time it has been accepted that algebra should be addressed from the early elementary grades given its potential to develop children's ability to analyse, represent and generalise mathematical relationships (NCTM, 2000; Suh, 2007).

While some subdomains of algebra, e.g. quadratic equations, are too advanced for young learners there are many algebraic ideas that are within their reach e.g. equality. One aspect of algebra that is suitable to teach to elementary school children is the concept of functions due to its potential to develop appropriate algebraic thinking (Reeves, 2005/06; Van de Walle, 2003). While a formal treatment of function does not occur until secondary school in the Irish education system, the concept of function can be introduced informally in the primary classroom. By definition, a function is "…a rule that uniquely defines how the first or independent variable affects the second or dependent variable" (Van de Walle, 2003, p. 436). Research has shown that this concept can be introduced in primary classrooms provided that adequate supports are provided to children when reasoning about this concept (Reeves, 2005/06; Suh, 2007). In order to make this concept of function accessible to children, the way in which the concept is introduced is crucial. In reality to teach algebra as most of us were taught would be potentially disastrous.

Mathematics education research plays a crucial role in investigating the effects of "artificial objects" on the development of mathematical understanding through the design and cyclical process of testing, reiteration and modification of objects (Wittman, 1998). When an emphasis is placed on both objects and instructional approaches, children can be supported in accessing powerful mathematical ideas. While many tools (e.g. technology) and approaches are used in primary classrooms to develop understanding of complex mathematical concepts, we briefly outline one approach we drew on in the design of lessons that supported primary children in gaining access to the concept of function: models. *Models* and *manipulatives* are well established in primary mathematics education particularly in relation to place value and number work. However, in order to be optimally effective, young children need support in making connections between the analogies embodied by the tools/manipulatives and the mathematical ideas, i.e., that the children use the models/manipulatives to represent the formal mathematics within the problem to be solved (Gravemeijer, 1999, p. 159). Popular approaches when teaching functions include the use of visual/geometric growing patterns, physical function machines and appropriate children's literature (e.g. *Two of Everything* by Lily Toy Hong (1993)). Function can be introduced in tandem with work on patterns and relations. However, physical function machines provide the potential to develop a conceptual understanding of function as it allows children to model the problem situation and analyse the change which comes about as a result of a function using simple tables (Reeves, 2005/06; Suh, 2007; Billings et al., 2007/08). Given that within the Irish context, various studies have reported unsatisfactory use of resources in the strand of Algebra (Shiel et al., 2006), the authors were anxious to explore the potential uses and impact of models/manipulatives within this mathematics strand.

This paper examines the mathematics content area of algebra; a domain traditionally considered to 'belong' to the domain of secondary level mathematics. It focuses in particular on the concept of function, describing the use of a specific tool to mediate learning. A physical function machine was used to provide a meaningful context and concrete vehicle to support children in developing conceptual understanding of function.

METHOD

Participants

This study was carried out with 21 final year pre-service primary teachers during the concluding semester of their teacher education program. Participants had completed their mathematics education courses (three semesters) and all teaching practice requirements (at junior, middle and senior grades) and self-selected into mathematics education as a cognate area of study. Four participants were male; the remainder were female. Two participants were international Erasmus students.

In the particular group working on the concept of function, there were five pre-service elementary teachers who worked alongside three mathematics educators to design a series of instructional activities to promote development of a deep understanding of function. Initial instructional activities developed were field-tested and refined through working with two different 4th class groups in different schools in Limerick city.

Lesson Study

All pre-service teachers, and three mathematics educators, engaged in *Japanese Lesson Study* which is an approach for studying teaching that utilizes detailed analyses of classroom lessons (Fernandez & Yoshida, 2004; Lewis, 2002; Lewis & Tsuchida, 1998). In this study, lesson study was used to examine the planning and the implementation of lessons in classrooms and thus facilitated the design of tools and sequences of instruction to support the development of algebraic reasoning with primary children. This paper reports on the work of one lesson study group - the 'Function' group.

The research was conducted over a 12-week semester. Participants worked collaboratively in groups of 5-6 on the design and implementation of a research lesson. While the first phase involved the *research and preparation* of a study lesson, i.e., researching the concept of function in order to construct a detailed lesson plan, the *implementation* stage involved one pre-service teacher teaching the lesson in a 4^{th} class primary classroom while the remainder of the group and the researchers observed and evaluated classroom activity and student learning. Subsequently, following discussion, the original lesson design was modified in line with their observations. The *second implementation* stage involved re-teaching the lesson with a second class of primary children and *reflecting* upon observations. The second implementation was videotaped. This cycle concluded with each lesson study group making a presentation of the outcomes of their work to their peers and lecturers at the end of the semester.

EXPLORING THE CONCEPT OF FUNCTION

This paper provides a detailed description of the teaching of the concept of function, with a particular focus on the tool used to facilitate the development of algebraic reasoning.

Setting the scene: The context of the 'Faulty Oven'

To launch the lesson, the teacher presented the scenario, "Yesterday I was baking. I have to admit I like baking." The children were immediately engaged with the context and eager to find out more. After asking the children to indicate whether they liked baking through a show of hands, the story continued with the teacher pointing to a picture of two cakes (see figure 1) stating, "I put these two cakes into my oven and something very strange happened. When I opened the oven door something had gone terribly wrong." Children were then invited to predict what might have happened. Students' responses were initially generic, e.g. "They were burnt" (Bernadette); "They were the wrong colour" (Callum); "Did they taste bad?" (Ryan). The teacher revealed that when the oven was opened there were not two but rather four cakes (see figure 2). On prompting, the children came up with mathematical explanations of what had happened:

Teacher: What do you think happened?

James: The oven added two. (+2)

Grace : The oven doubled them. (x2)

Joseph: Maybe they used self-raising flour?

Figure 1: Picture of cakes placed in oven

Figure 2: Picture of cakes taken out of oven

Additional scenarios were presented to provide children with further opportunities to predict and identify the nature of the function, e.g. “Later I made five lasagnes. Can anyone guess what happened when I opened my oven?” Children made predictions (e.g. seven lasagnes (rule: +2), ten lasagnes (rule: x2 or double)). Again children were presented with the image of the lasagnes which came out of the oven (10 lasagnes). By working through this second example, children tested predictions and made a decision regarding the ‘rule’ being used. When the class was asked what was wrong with the oven, a student Keith, responded “It’s a cloning machine.” This statement highlighted that the children had quickly grasped the concept of function. Through the use of probing questions such as “What happened to the food each time?’ and further scenarios (e.g. “If I then made three apple pies…”) students successfully identified the rule e.g. “it multiplied” (Ross); “it doubled” (Joseph).

Subsequently children were informed that the oven was a magic machine called a “function machine”. Some participating teachers chose “math rule maker” as an alternative label, believing that it provided guidance to children regarding the working of the machine (“So it makes math rules”), thus making the concept of function more accessible. The physical ‘faulty oven’ function machine (an aluminium foil covered box with coloured buttons) was revealed to the class (see figure 3) and generated great mystery and amusement amongst children. The following excerpt demonstrates how the features of the function machine (input, output, and rule buttons) were introduced.

Figure 3: Faulty Oven

Teacher: So this is the faulty oven we use to make math rules. It has two doors. We’ve got a little door here. This is where we put in our food.

(Teacher points to the input drawer, see figure 4)

There is another door at the back where the food comes out.

(The teacher opens the output door, see figure 5)

	Anything we put in here is the 'input'. (*Pointing to the input door*)
	Why do you think we call it the 'input'?
Cian:	It's what you put in.
Teacher:	It's what you put in it, excellent. So 'put' and 'in' – 'input'… What do you think we call what comes out?
Lorna:	The output?
Teacher:	The output- in and out. It's easy to remember isn't it? Now do you see these buttons down along the side of the machine.
	(The teacher points to the front of the machine, see figure 3)
	What do you think they do?
Louis:	Am …that's what makes them multiply.
Teacher:	That's what makes them multiply-ok- and would each button do the same thing?
Bernadette:	They could add, multiply, divide or minus.
Teacher:	Excellent, so they might all have different…?
Children:	Rules.

One child subsequently volunteered his interpretation of the various parts of the machine

Robert:	The button is a switch and the output is what the switch activates.

This thinking was quite sophisticated and suggests that the function machine context provided a relevant concrete vehicle to facilitate children in developing conceptual understanding of function.

Using the 'Faulty Oven' function machine

The focus then moved to children experiencing the math rule maker in action. Base ten materials (e.g. Dienes' blocks) were used to represent the food (input and output). All inputs and outputs were prepared in advance in plastic trays (see figures 4 and 5). While children could see and examine input trays, output trays were hidden (unknown to children) in a secret compartment in the back of the function machine (see figure 5).

A child was invited to the top of the class (see figure 6). He/she counted the number of Dienes' blocks in the pre-prepared input tray (e.g. 7 cubes), placed the tray in the input slot and pressed the appropriate (function) button, e.g. blue. The teacher, not the child, then opened the slot at the back of the function machine (figure 5), discretely hid the input tray and selected the appropriate output tray (figure 6). On handing this tray to the child, the child counted the output (e.g. 11 cubes). Children were not aware of the exchange of trays which took place in the function machine and were surprised when the output tray had a different number of cubes than the input tray. Led by the teacher, the class recorded the 3-4 inputs and outputs for each selected button in an effort to predict the rule. While the teacher used a poster to record inputs and outputs for the selected button (see figure 6), children recorded on a pre-prepared worksheet.

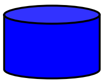
Figure 4: Input drawer and tray

Figure 5: Back view of Faulty Oven (output drawer and trays)

Figure 6: Using the function machine

Figure 7: Teacher-led activity (orange button)

On presenting a number of related inputs and outputs, the teacher gave the class time to work out the possible rule/function (e.g. blue rule could be +16 or x9 after the first input/output- see figure 8).



Blue Button		
Input	**Output**	**Rule**
2	18	
23	39	

Figure 8: Recording the inputs/outputs for the blue button

To promote success, initial rules for the buttons focused on one-step rules (e.g. “add five” or “double”.). However, the level of challenge increased incrementally culminating with the final button (yellow in this case) which consisted of a more complex 2-step rule, e.g. “multiply by two and add three.” When being introduced to two-step rules, children were informed that all rules so far were one-step rules, e.g. x2, +15, x9. They were invited to provide an example of a two-step rule. We found there was a tendency for children to provide examples such as “+3+5” (Monica) and “+25-20” (William). The teacher highlighted that such examples e.g. +3+5 could be described as a one-step rule i.e. +8. Children also had the opportunity to work out the input when given output i.e. work backwards and justify their approach, e.g. “If the output is fifty-four, what is the input?...how did you know that?” Subsequently, children gave examples of two-step rules. The following excerpt illustrates the nature of the dialogue which took place in one classroom when working out a two-step rule.

Teacher: Our input was?

Pavol: Seven

Teacher: …and our output is?

Class: Seventeen

Teacher: Now if that was a one-step rule what could the rule be? Emma?

Emma: Add ten

Teacher: Add ten

But it’s a two-step rule so there are two steps in it. O.K.

	(*The teacher gestures to William to come help with the next input/output for the yellow button*)
	So what's our input this time?
William:	Eight
Teacher:	Put it (*Refers to the Dienes' blocks*) in and press the yellow two-step button.
	(*William places the tray into the input drawer and presses the yellow button. The teacher discreetly selects the appropriate output tray from the back of the machine and presents it to William*)
	What is the output?
William:	Nineteen.
Teacher:	Your input was eight and your output was nineteen. What could the rule be? It could be adding - what else could it be doing?
Monica:	Multiplying.
Teacher:	Multiplying and adding. Has anyone any other idea?
	(*Students given some time to think, some children start to raise their hands*)
	Ok I am going to give you a hint: the first step is 'multiplied by two'. So we multiply it by two first, then we do something else to it. Have another look at it now.
	(*The teacher circulates and observes while the children work and raise their hands*)
	Lorna, what did you get?
Lorna:	Times two plus three?

Independent work

Once the four rules had been identified, children then worked in groups to generate and identify their own rules/functions. Groups were provided with record sheets, these sheets reflected many of the features of the original function machine – they used the same language (input/output) and contained the same perceptual cues to identify rules/functions (coloured buttons similar to that found on the function machine). Each group member took turns to act as the function machine. This involved the 'math rule maker' child secretly constructing a rule (e.g. +5). She then received an input (between 0 and 10) from each group member and calculated the corresponding outputs (figure 9). Children were encouraged to use multiplication and addition only in their rules, given that rules involving division and subtraction may lead to outputs being fractions or negative numbers. Children were encouraged to create a two-step rule if they wished. Each group member recorded all the relevant information (inputs, outputs) in order to promote prediction and checking. The rules which students generated ranged from "plus five" to "times seven" to "times nine plus one". Children experienced little difficulty making and testing predictions regarding the rule. Two-step rules proved more challenging in this regard, and required children to demonstrate perseverance when using strategies such as trial and error, guessing and checking.

Figure 9: Pupils taking turns giving inputs to the math rule maker

As a concluding activity, children were given the opportunity to share their rule with the class. While only one example of a two-step rule or function was taught during the lesson, the independent work highlighted that teaching mathematics using effective models and representations helps develop sophisticated mathematical reasoning and understanding among children. The next excerpt reflects the reality in many classes where the complexity of some of the children's rules was a source of amazement for the teacher as well as a source of genuine challenge for the children.

Teacher: Robert's rule is so complicated it only works for even numbers.

Teacher: Can you give Robert an input? Remember it could be division or subtraction in this rule. It's a two-step rule, is it?

Robert: Yeah.

Teacher: Can we have an input?

Josh: Two.

Teacher: Two. Robert, what's your output?

(Robert writes '7' as the related output on the board)

We'll take another input.

Callum: Ten

(Robert writes '11' as the related output on the board)

Teacher: O.K., David?

David: Four.

(*Robert refers to his worksheet and then writes '8' as the related output on the board*. *Students are given some time to think and work*)

Robert: The first step is addition and the second step is division.

(*Students given further time to think and work, additional students raise their hands*)

Teacher: Another hint: the first step is 'plus twelve'. O.K., and the second step is division. So what you will have to do is imagine adding twelve to our input and try and figure it out then.

Laura: Plus twelve divided by two.

Teacher: Plus twelve divided by two. Is she right?

Class: Yeah.

IN CONCLUSION

The development and use of the tool described in this study were critical in supporting relatively young children in reasoning about functions. The research indicates that functions can be taught to primary school pupils and while these were children were taught by relatively in-experienced (pre-service) teachers, we believe that the design of the physical function machine which facilitated the use of concrete materials; alongside the selection of the context which the children could relate to; meant that the concept of function was accessible to these relatively young children. The function machine acted as an effective tool in promoting conceptual understanding of function among children in the participating schools, where all of the children engaged in the lessons and developed the appropriate conceptual understandings. In fact some of the children exceeded the expectations.

This research carried out in classrooms also shows that understanding functions can be fun! The use of the 'faulty oven' function machine led to much enjoyment. While the creation of a function machine requires effort in sourcing and assembling the component parts, once made, the function machine is an invaluable resource. Beyond algebra, the function machine can be used to promote understanding of operations and number facts. We found that due to the novel appearance and unconventional working of the function machine, children were interested and excited in exploring and creating functions, and the classrooms we visited were buzzing with questions and predictions. However as is evident from this paper, using the function machine does not mean reducing the mathematics challenge provided to children. In short, the approach has the potential to promote students' motivation, enjoyment, involvement and most importantly understanding of the concept of function.

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