

MARKOVIANITY AND THE THOMPSON MONOID F^+

CLAUS KÖSTLER, ARUNDHATHI KRISHNAN, AND STEPHEN J. WILLS

Dedicated to the memory of Vaughan Jones

ABSTRACT. We introduce a new distributional invariance principle, called ‘partial spreadability’, which emerges from the representation theory of the Thompson monoid F^+ in noncommutative probability spaces. We show that a partially spreadable sequence of noncommutative random variables is adapted to a local Markov filtration. Conversely we show that a large class of noncommutative stationary Markov sequences provides representations of the Thompson monoid F^+ . In the particular case of a classical probability space, we arrive at a de Finetti theorem for stationary Markov sequences with values in a standard Borel space.

CONTENTS

1	Introduction and some main results	2
2	Preliminaries	10
	2.1 The Thompson group F and its monoid F^+	10
	2.2 The partial shifts monoid S^+	11
	2.3 Noncommutative probability spaces and Markov maps	11
	2.4 Noncommutative random variables and distributional invariance principles	12
	2.5 Noncommutative independence and Markovianity	14
	2.6 Noncommutative stationary processes and dilations	18
3	A new distributional invariance principle in classical probability theory	24
	3.1 Algebraization of classical probability theory	24
	3.2 Partial spreadability and de Finetti theorems in classical probability	25
	3.3 Functions of classical stationary Markov sequences and their algebraization	31
	3.4 A result on Markov dilations in classical probability theory	32
4	Markovianity from representations of the Thompson monoid F^+	37
	4.1 Representations with a generating property	38

2020 *Mathematics Subject Classification*. Primary: 46L53, 60J05; Secondary: 60G09, 20M30.

Key words and phrases. Distributional Invariance Principles; Noncommutative De Finetti Theorems; Noncommutative Stationary Markov Processes; Representations of Thompson monoid F^+ .

4.2	Commuting squares and Markovianity for shifted fixed point algebras	40
4.3	Commuting squares and Markovianity for stationary processes	43
4.4	A noncommutative version of de Finetti's theorem	47
5	Construction of representations of the Thompson monoid F^+	48
5.1	Tensor product constructions	48
5.2	Constructions in classical probability	53
5.3	Constructions in the framework of operator algebras	55
	Acknowledgements	58
	References	59

1. INTRODUCTION AND SOME MAIN RESULTS

The goal of this paper is to introduce ‘partial spreadability’ as a new distributional invariance principle as it emerges from the representation theory of the Thompson monoid F^+ and to present first results on the following discovery: the monoid F^+ algebraically encodes Markovianity in an operator algebraic framework of noncommutative probability. Here the attribute ‘noncommutative’ is understood in the sense of ‘not-necessarily-commutative’, and thus applies to classical probability, free probability and, more generally, quantum probability. We limit our considerations in this paper to representations of the Thompson monoid F^+ which connect to unilateral Markov shifts and just note that, with some more effort, representations of the Thompson group F can be seen to connect to bilateral Markov shifts [54]. Further generalizations of our approach to noncommutative random fields are possible and of interest for future investigations.

Notably, our results are also applicable to subfactor theory where independently, Vaughan Jones constructed and studied representations for the Thompson group F , within the ongoing effort to obtain a conformal field theory for every finite index subfactor [46, 47, 48, 9, 10, 3]. How Vaughan Jones’ approach and our approach are related is of high interest to be further investigated, but beyond the scope of the present paper. Furthermore, our approach invites further investigations of the representation theory of the Thompson group F , and its relatives, from the probabilistic viewpoint suggested by our new approach.

To increase the accessibility of our approach, let us focus in this introduction first on classical probability where distributional symmetries and invariance principles are known to provide deep structural results on stochastic processes [52]. Its subject of research emerged around the early 1930s when de Finetti characterized an exchangeable infinite sequence of $\{0, 1\}$ -valued random variables as a mixture of independent identically distributed (i.i.d.) random variables. This foundational result was extended to random variables with values in a compact Hausdorff space by Hewitt and Savage [45], and to values in a Borel space by Aldous [4]. Furthermore, Ryll-Nardzewski [67] established that the apparently weaker distributional symmetry of spreadability (also known as ‘contractability’ in the literature) is equivalent to exchangeability for infinite sequences.

Already de Finetti considered ‘partial exchangeability (of the Markov type)’ or ‘Markov exchangeability’, suggesting that this distributional symmetry should characterize a mixture of infinite sequences of Markov chains (see also [27]). Initially Freedman [36] established such a characterization for stationary sequences, taking values in a countable set. Consecutively, Diaconis and Freedman relaxed the stationarity condition to that of recurrence, to arrive in [26] at a de Finetti theorem for infinite Markov chains. In particular, they show that partial exchangeability can be implemented by ‘block-switch transformations’ on the path space of the stochastic process. A different and more sophisticated approach was taken by Kallenberg for sequences of random variables with values in an arbitrary measurable space [50]. Invoking stopping time arguments, he characterized recurrent ‘locally \mathcal{F} -homogeneous’ sequences to be ‘conditionally \mathcal{F} -Markovian’ (see [50, Theorem 2.4]). Furthermore, for random variables with values in finite or countable sets, de Finetti-type characterization results in terms of hidden Markov models are also available for certain exchangeable sequences [23, 24, 25] and, more recently, for certain partially exchangeable sequences [34, 35].

Quite recently, it was realized in [31] that actually not exchangeability, but spreadability is the fundamental distributional invariance principle from the viewpoint of algebraic topology and homological algebra. This insight is little apparent from the common definition of spreadability which states that joint distributions of a sequence of (noncommutative) random variables are invariant upon passing to a subsequence (see Definition 2.4.1). Indeed, [31, Theorem 1.2] establishes a new equivalent characterization of spreadability which connects it to the representation theory of the ‘partial shifts monoid’

$$S^+ := \langle h_0, h_1, h_2, \dots \mid h_k h_\ell = h_{\ell+1} h_k \text{ for } 0 \leq k \leq \ell < \infty \rangle^+.$$

Roughly phrasing, S^+ algebraically encodes conditional independence in classical probability. More precisely – in fact, this can be taken as an alternative definition – a sequence of random variables $\xi \equiv (\xi_n)_{n \geq 0}$ is spreadable if and only if there exists a representation ϱ_* of S^+ in the measure-preserving measurable maps of the underlying probability space such that

$$\xi_0 = \xi_0 \circ \varrho_*(h_n), \quad \xi_n = \xi_0 \circ \varrho_*(h_0^n)$$

for all $n \geq 1$. This adds to the extended de Finetti theorem a new equivalent characterization of conditionally independent, identically distributed random variables (see [31, Theorem 1.2], which is partly stated as Theorem 3.2.1 for the convenience of the reader).

This alternative definition of spreadability deserves some elaboration. Commonly, a probabilistic approach implements distributional symmetries and invariance principles in terms of actions on the index set of a sequence of random variables. For example, exchangeability requires that the distribution of the sequence is invariant under permuting the indices of the sequence, and spreadability requires that the distribution is invariant under passing to a subsequence. Here we take the alternative point of view that the distributional invariance principle is implemented through a group or a monoid acting on the underlying probability space, as illustrated above for spreadability by the partial shifts monoid S^+ . In fact, such a point of view initiated the development of ‘braidability’ in [40] as a new distributional symmetry which is intermediate to exchangeability

and spreadability in noncommutative probability, and this was further investigated for braided parafermions in [11]. Also, the introduction of quantum exchangeability in [55] was inspired by this point of view when replacing the action of permutations by co-actions of quantum permutations.

This alternative definition of spreadability through measure-preserving measurable actions of S^+ on a probability space is the starting point of our investigations on possible generalizations of this distributional invariance principle. It is an elementary observation that the monoid S^+ is a quotient of the Thompson monoid

$$F^+ = \langle g_0, g_1, g_2, \dots \mid g_k g_\ell = g_{\ell+1} g_k \text{ for } 0 \leq k < \ell < \infty \rangle^+,$$

as relations of the form $g_k g_k = g_{k+1} g_k$ are absent in the presentation of F^+ .¹ Thus it is intriguing to ask if the representation theory of F^+ is connected to a novel distributional invariance principle which characterizes a larger class of random objects than those characterized by spreadability. In other words:

Can one characterize a ‘partially spreadable’ sequence of random variables ξ ?

Here ‘partial spreadability’ means by definition that the sequence $\xi \equiv (\xi_n)_{n \geq 0}$ satisfies

$$\xi_0 = \xi_0 \circ \rho_*(g_n), \quad \xi_n = \xi_0 \circ \rho_*(g_0^n)$$

for all $n \geq 1$, where now ρ_* denotes a representation of the Thompson monoid F^+ in the measure-preserving measurable maps on the underlying probability space, as introduced in Definition 3.2.2. We give an affirmative answer to this question in the following theorem, the proof of which is given in Subsections 3.2 and 5.2, where (Ω_0, Σ_0) denotes a standard Borel space.

Theorem 1.0.1. *Let $\xi \equiv (\xi_n)_{n \in \mathbb{N}_0}$ be a sequence of (Ω_0, Σ_0) -valued random variables on the standard probability space (Ω, Σ, μ) . Then the following are equivalent:*

- (a) ξ is maximal partially spreadable;
- (b) ξ is a stationary Markov sequence.

It is worthwhile to point out for the implication ‘(a) \Rightarrow (b)’ that, in general, a partially spreadable sequence may fail to be Markovian. This motivates the introduction of the strengthened notion of ‘maximal partial spreadability’ in Definition 3.2.2. This maximality condition requires that the random variable ξ_0 generates a σ -subalgebra of Σ which equals the intersection of all μ -almost $\rho_*(g_n)$ -invariant sets for $n \geq 1$. It takes into account the fact that a function of a stationary Markov sequence is still partially spreadable (see Theorem 3.3.2), but may be non-Markovian.

The proof of the converse implication ‘(b) \Rightarrow (a)’ requires us to find a representation of the Thompson monoid F^+ which cannot be easily identified when the underlying probability space is given by the Daniell-Kolmogorov construction for the Markov process. But a natural choice for such a representation exists when the Markov process is written as an open dynamical system. Such an approach is common for the construction of quantum Markov processes, as pioneered in [1]. Actually, our setting requires an operator algebraic approach to noncommutative stationary Markov processes as presented

¹This definition of the Thompson monoid F^+ differs from the usual one used in the literature (see also Subsection 2.1). Our choice of presentation is motivated from the elementary observation that the monoid S^+ is a quotient of F^+ .

by Kümmerer [56, 57] and used by Anantharaman-Delaroche [5], as well as Haagerup and Musat [44]. Furthermore, we refer the interested reader to Attal's expository paper [6] for detailed information on the open dynamical system viewpoint for (possibly non-stationary) classical Markov sequences.

Having exemplified by Theorem 1.0.1 how Markovianity and representations of the Thompson monoid F^+ relate in classical probability, next we turn our attention to developing noncommutative versions of this relation. In parts, our approach is stimulated by the significant progress made over the past decade in establishing noncommutative de Finetti type results in the context of free independence [18, 55, 19, 20, 7, 21, 28, 37, 29, 30], Boolean or monotone independence [61, 62, 63, 16], fermionic systems [11, 14, 17, 32, 33], and general operator algebraic settings [2, 40, 41, 53, 42, 15, 31].

An immediate idea is to look for a direct transfer of the de Finetti theorem of Diaconis and Freedman [26] or of Kallenberg's characterization in [50] to a noncommutative setting. Such a direct route presently does not seem to be available, as the first approach requires the availability of certain pathwise transformations and the latter utilizes stopping time arguments. But combining our approach to distributional invariance principles and the open dynamical system formulation of stationary Markov processes, we are able to suitably transfer Theorem 1.0.1 to noncommutative probability.

We now address our de Finetti type results obtained in the context of 'partial spreadability' in an operator algebraic setting. We refer the reader to Section 2 for definitions and notation as we will use them below. Here we just remind that the pair (\mathcal{M}, ψ) denotes a (noncommutative) probability space which consist of a von Neumann algebra \mathcal{M} and a faithful normal state ψ on \mathcal{M} . Such pairs are also known as W^* -algebraic probability spaces in the literature. Furthermore, a noncommutative random variable ι_0 from the probability space (\mathcal{A}, φ) into the probability space (\mathcal{M}, ψ) is given by an injective $*$ -homomorphism $\iota_0: \mathcal{A} \rightarrow \mathcal{M}$ such that $\psi \circ \iota_0 = \varphi$, and written as $\iota_0: (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$. (Here we have omitted addressing a modular condition as done in Subsection 2.4.) Finally $\text{End}(\mathcal{M}, \psi)$ denotes certain unital $*$ -homomorphisms on \mathcal{M} , as further detailed in Definition 2.3.1.

Let us comment on our choice of a W^* -algebraic setting where von Neumann algebras are equipped with faithful normal states. Instead, considering unital C^* -algebras with states as a starting point may seem more appropriate in the context of noncommutative de Finetti theorems. For example, Størmer characterized in [68] symmetric states on the infinite (minimal) tensor product of a unital C^* -algebra with itself. As another more recent example, quantum symmetric states were characterized in [29] on the infinite universal free product of a unital C^* -algebra with itself. Even though these free or tensor product constructions provide noncommutative versions of the Daniell-Kolmogorov construction, they give little insight on how to associate a representation of the Thompson monoid F^+ to a noncommutative stationary Markov process, for the same reasons as in the classical theory. As the primary goal of the present paper is to establish the connection between Markovianity and 'probabilistic' representations of F^+ , we chose von Neumann algebras with faithful normal states as our starting point.

This avoids an additional technical overhead, to improve the transparency of our new approach, and postpones C*-algebraic approaches to future publications.

To provide some context to our main results, let us first recall the following definition of spreadability as a distributional invariance principle which was identified in [53, 31] to be equivalent to its more traditional formulation (as stated in Definition 2.4.1(ii)).

Definition 1.0.2 ([53, 31]). A sequence of random variables $\iota \equiv (\iota_n)_{n \geq 0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ is *spreadable* if there exists a representation $\varrho : S^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ such that the following localization and stationarity properties are satisfied:

$$\iota_0 = \varrho(h_n)\iota_0 \quad \text{for all } n \geq 1; \quad (\text{L})$$

$$\iota_n = \varrho(h_0^n)\iota_0 \quad \text{if } n \geq 0. \quad (\text{S})$$

More generally, ι is said to be *spreadable* if there exists a spreadable sequence $\tilde{\iota}$ such that $\iota \stackrel{\text{distr}}{=} \tilde{\iota}$.

Replacing the role of the partial shift monoid S^+ by the Thompson monoid F^+ , we are now in the position to introduce a natural generalization of spreadability as a new distributional invariance principle.

Definition 1.0.3. A sequence of random variables $\iota \equiv (\iota_n)_{n \geq 0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ is *partially spreadable* if there exists a representation $\rho : F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ such that the following localization and stationarity properties are satisfied:

$$\iota_0 = \rho(g_n)\iota_0 \quad \text{for all } n \geq 1; \quad (\text{L})$$

$$\iota_n = \rho(g_0^n)\iota_0 \quad \text{if } n \geq 0. \quad (\text{S})$$

More generally, ι is said to be *partially spreadable* if there exists a partially spreadable sequence $\tilde{\iota}$ such that $\iota \stackrel{\text{distr}}{=} \tilde{\iota}$.

Clearly spreadability implies partial spreadability. Of course, the crucial question is if partial spreadability allows to develop similar results of de Finetti type as it is the case for spreadability, especially in the general framework of noncommutative probability. As detailed in [53], conditional independence in classical probability is generalized to a geometric notion which we call here ‘conditional CS independence’ (see Definition 2.5.5) and which is intimately related to Popa’s notion of commuting squares in subfactor theory. This geometric viewpoint on a very general notion of noncommutative independence emerged out of the investigations of Kümmerer on the structure of noncommutative stationary Markov processes [56] and is further justified by the following noncommutative extended de Finetti theorem.

Theorem 1.0.4 ([53]). *Let $\iota \equiv (\iota_n)_{n \geq 0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ be a sequence of random variables and consider the following conditions:*

- (a) ι is spreadable;
- (b) ι is stationary and conditionally CS independent;
- (c) ι is identically distributed and conditionally CS independent.

Then one has the following implications:

$$(a) \implies (b) \implies (c).$$

We refer the interested reader to the introduction of [53] and to [40, 42] to learn more on why one should not expect an equivalence of these three statements in the general framework of noncommutative probability, in contrast to the situation of classical probability or free probability. Replacing spreadability by partial spreadability we have succeeded to establish the following main result of de Finetti type.

Theorem 1.0.5. *Let $\iota \equiv (\iota_n)_{n \geq 0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ be a sequence of random variables and consider the following conditions:*

- (a) ι is partially spreadable;
- (b) ι is stationary and adapted to a local Markov filtration;
- (c) ι is identically distributed and adapted to a local Markov filtration.

Then one has the following implications:

$$(a) \implies (b) \implies (c).$$

Here the notion of ‘adaptedness to a local Markov filtration’ is cast in Definition 2.5.10. The proof of this main result is the subject of Subsection 4.4. Clearly, the converse implication from (c) to (b) fails for the obvious reason that an identically distributed sequence may not be stationary. Also one should not expect that (b) implies (a) in the general noncommutative setting, for similar reasons of the corresponding failure in Theorem 1.0.4. At the time of this writing we do not know if the conditions (a) and (b) are equivalent in the commutative setting corresponding to classical probability.

Commonly Markovianity is introduced in the literature as a property of a stochastic process with respect to a filtration. Our approach is slightly more general in two aspects. Similar to random fields, our approach considers ‘local filtrations’ which are *partially ordered* families of von Neumann subalgebras, or σ -subalgebras, indexed by ‘discrete time intervals’. Furthermore these ‘local filtrations’ allow us to introduce Markov properties directly, *without* reference to random variables (see Definition 2.5.7 and Definition 3.2.4, respectively).

Let us stress that, as already stated in the setting of classical probability (see Subsection 3.3), a partially spreadable sequence may fail to be a Markov sequence. (Loosely phrased, this is connected to the fact that commuting square properties of von Neumann subalgebras are not robust under restrictions, compare Remark 3.2.5.) But Markovianity can be enforced by strengthening ‘partial spreadability’ to ‘maximal partial spreadability’ (as done in Definition 4.4.2). Normally the random variable ι_0 of a partially spreadable sequence ι satisfies only the localization property

$$\iota_0(\mathcal{A}) \subset \bigcap_{n \geq 1} \mathcal{M}^{\rho(g_n)},$$

where $\mathcal{M}^{\rho(g_n)} = \{x \in \mathcal{M} \mid \rho(g_n)(x) = x\}$ denotes the fixed point von Neumann subalgebra of the represented generator $\rho(g_n)$. This inclusion condition is now strengthened to the equality

$$\iota_0(\mathcal{A}) = \bigcap_{n \geq 1} \mathcal{M}^{\rho(g_n)}$$

for the definition of a maximal partially spreadable sequence. Consequently, one arrives at Theorem 4.4.3 which, for the convenience of the reader, we state here slightly

reformulated and aligned with the formulation of the classical de-Finetti-type result in Theorem 1.0.1:

Theorem 1.0.6. *Let ι be a sequence as in Theorem 1.0.5 and consider the following conditions:*

- (a) ι is maximal partially spreadable;
- (b) ι is a stationary Markov sequence.

Then the implication (a) \implies (b) is valid.

Again, one cannot expect the converse implication to be true in the full generality of the noncommutative setting, as reminded above already.

We are left to outline the content of this paper. Most of our main results are proven in an operator algebraic framework of noncommutative probability, as introduced in Section 2. We provide definitions, notation and some background results on the Thompson monoid F^+ in Subsection 2.1, and on the partial shift monoid S^+ in Subsection 2.2. The basics of noncommutative probability spaces and Markov maps are provided in Subsection 2.3. Subsection 2.4 is devoted to noncommutative random variables and established distributional invariance principles like exchangeability, spreadability and stationarity. We present in Subsection 2.5 a geometric notion of noncommutative conditional independence which essentially resembles a structure known as commuting squares in subfactor theory. Furthermore we provide the notion of a local Markov filtration which allows to define Markovianity on the level of von Neumann subalgebras without any reference to noncommutative random variables. Actually these structures underly Kümmerer’s approach to stationary Markov dilations and can now be seen to emerge from distributional invariance principles in noncommutative probability. Finally we provide some results on noncommutative stationary processes in Subsection 2.6. Here we will meet the one-sided version of noncommutative stationary Markov processes and Markov dilations in the sense of Kümmerer [56] and as they are considered within the context of factorizable Markov maps by Musat and Haagerup [44].

Section 3 is devoted to the introduction of ‘partial spreadability’ as a new distributional invariance principle in classical probability theory. Markovianity and representations of the Thompson monoid F^+ are addressed in the traditional language of classical probability. Subsection 3.1 sketches how one arrives at noncommutative notions of probability spaces and random variables when starting from a traditional setting. Subsection 3.2 presents the extended de Finetti theorem, Theorem 3.2.1, to create some relevant background for the introduction of the new distributional invariance principles of ‘partial spreadability’ and ‘maximal partial spreadability’ in Definition 3.2.2. Furthermore we introduce in Definition 3.2.4 the notion of a local Markov filtration which is slightly more general than those usually considered for Markov sequences, but quite familiar in the topic of random fields or generalized stochastic processes. This allows us to formulate several de-Finetti-type results for (maximal) partially spreadable sequences, in particular Theorem 3.2.12 and Theorem 1.0.1, our main results of de Finetti type on the characterization of stationary Markov sequences in classical probability. We continue in Subsection 3.3 with providing evidence that the class of partially spreadable sequences is much larger than the class of (mixtures of) stationary Markov sequences.

For this purpose, we discuss functions of stationary Markov chains, in particular in the context of the algebraization procedure. This yields our main result in Theorem 3.3.2 that functions of partially spreadable sequences remain partially spreadable (but may no longer be Markovian, as it is well-known). Finally, we provide in Subsection 3.4 a general result on stationary Markov processes as open dynamical systems, Theorem 3.4.2. It builds on the usual Daniell-Kolmogorov construction for such processes and underpins Kümmerer's open dynamical systems approach to stationary Markov processes [56]. Our presentation follows closely the arguments given by Kümmerer in [57]. In particular, we will illustrate this approach by zero-one-valued stationary unilateral Markov sequences in Example 3.4.3. These results will be drawn upon in Subsection 5.2 for the construction of a representation of the Thompson monoid F^+ . Effectively, the availability of such an alternative construction entails that a stationary Markov process is (maximal) partially spreadable. This establishes one direction of the claimed equivalence in the de Finetti theorem, Theorem 1.0.1.

Section 4 investigates representations of the Thompson monoid F^+ in the endomorphisms of noncommutative probability spaces. Subsection 4.1 introduces the generating property of representations of F^+ in Definition 4.1.1. This property ensures that the fixed point algebras of the represented generators of F^+ form a tower which generates the noncommutative probability space, see Proposition 4.1.5. Effectively this tower of fixed point algebras equips the noncommutative probability space with a filtration which, using actions of the represented generators, can be further upgraded to become a local Markov filtration. We show in Subsection 4.2 that 'shifted' fixed point algebras of the represented generators of F^+ provide triangular towers of commuting squares. Local Markov filtrations are obtained from this as a particular property, see Corollary 4.2.5. Subsection 4.3 considers certain noncommutative stationary processes which are partially spreadable by construction. A main result is Theorem 4.3.4 which establishes that such processes are adapted to a local Markov filtration. Finally, the proof of the de Finetti theorem for stationary Markov processes, Theorem 1.0.5, is completed in Subsection 4.4.

Section 5 provides some elementary constructions of representations of the Thompson monoid F^+ . Here we focus on tensor product constructions in Subsection 5.1 and meet structures which are familiar from so-called tensor dilations of Markov operators [56, 57]. When specializing these tensor product constructions to commutative von Neumann algebras, we obtain representations of the Thompson monoid F^+ in the setting of classical probability. This is the subject of Subsection 5.2, where we will also complete the proof of the de Finetti theorem for stationary Markov sequences in the classical setting, Theorem 1.0.1. Finally, we turn our attention in Subsection 5.3 to constructions in the general framework of operator algebras. We introduce certain monoidal extensions of F^+ and S^+ , and investigate their representation theory, to adapt and refine Kümmerer's approach on noncommutative Markov processes as perturbations of noncommutative Bernoulli shifts [58].

2. PRELIMINARIES

2.1. The Thompson group F and its monoid F^+ . The Thompson group F , originally introduced by Richard Thompson in 1965 as a certain group of piece-wise linear homeomorphisms on the interval $[0, 1]$, is known to have the infinite presentation

$$F = \langle g_0, g_1, g_2, \dots \mid g_k g_\ell = g_{\ell+1} g_k \text{ for } 0 \leq k < \ell < \infty \rangle.$$

We note that these generators g_k of the group F correspond to the inverses of generators usually used in the literature (e.g. [8]). We work throughout with this group presentation as we are interested in the study of the following monoid. As the defining relations of this presentation involve no inverse generators, one can associate to it the monoid

$$F^+ = \langle g_0, g_1, g_2, \dots \mid g_k g_\ell = g_{\ell+1} g_k \text{ for } 0 \leq k < \ell < \infty \rangle^+, \quad (2.1.1)$$

henceforth referred to as the *Thompson monoid* F^+ . (We refer the reader to Remark 2.1.3 on how this monoid relates to the one usually introduced in the literature.)

An element $e \neq g \in F^+$ has the unique normal form

$$g = g_k^{a_k} g_{k-1}^{a_{k-1}} \cdots g_1^{a_1} g_0^{a_0}, \quad (2.1.2)$$

where $a_0, a_1, \dots, a_{k-1}, a_k \in \mathbb{N}_0$ with $a_k > 0$ for some $k \in \mathbb{N}_0$ (see [8, Theorem 2.4.4], for example, bearing in mind that we use the relations of the inverses).

Definition 2.1.1. Let $m, n \in \mathbb{N}_0$ with $m \leq n$ be fixed. The (m, n) -*partial shift* $\text{sh}_{m,n}$ is the monoid endomorphism on F^+ defined by

$$\text{sh}_{m,n}(g_k) = \begin{cases} g_m & \text{if } k = 0 \\ g_{n+k} & \text{if } k \geq 1. \end{cases}$$

Each map $\text{sh}_{m,n}$ is well-defined as a monoid endomorphism as it preserves all defining relations of F^+ . In particular, the endomorphism $\text{sh}_{1,1}$ satisfies $\text{sh}_{1,1}(g_k) = g_{k+1}$ for all $k \geq 0$. This shift is also considered in [8, Definition 1.6.5].

Lemma 2.1.2. The monoid endomorphisms $\text{sh}_{m,n}$ on F^+ are injective for all $m, n \in \mathbb{N}_0$.

Proof. Let $g \in F^+$ have the (unique) normal form as stated in (2.1.2). Then

$$\text{sh}_{m,n}(g) = g_{n+k}^{a_k} g_{n+k-1}^{a_{k-1}} \cdots g_{n+1}^{a_1} g_m^{a_0}$$

is again in normal form. The injectivity of the map $\text{sh}_{m,n}$ is concluded from the uniqueness of the normal form. \square

Remark 2.1.3. We emphasize that F^+ differs from the monoid usually considered in the literature, due to our choice of generators for F . Nevertheless, this monoid F^+ is left and right cancellative, and any two elements of F^+ admit a left common multiple. Thus F^+ can be identified with a monoid in the Thompson group F such that any element g of F can be written as $g = s^{-1}t$, with $s, t \in F^+$. We refer the reader to [22] for further details on related constructions. Finally, we remark that, alternatively, the generators of the monoid F^+ can be obtained as morphisms (in the inductive limit) of the category of finite binary forests, similarly as done in [8, 12, 47], for example.

2.2. The partial shifts monoid S^+ . Stipulating additional relations to those for the generators in (2.1.1), one obtains the monoid

$$S^+ = \langle h_0, h_1, h_2, \dots \mid h_k h_\ell = h_{\ell+1} h_k \text{ for } 0 \leq k \leq \ell < \infty \rangle^+$$

as a quotient of the Thompson monoid F^+ . The monoid S^+ is *not* a submonoid of the group with infinite presentation $\langle h_0, h_1, h_2, \dots \mid h_k h_\ell = h_{\ell+1} h_k \text{ for } 0 \leq k \leq \ell < \infty \rangle$, as the latter is isomorphic to the additive group \mathbb{Z} . Actually the generators h_k of the monoid S^+ satisfy relations as they are familiar for coface maps in simplicial cohomology. As discussed in [31], these generators arise as morphisms in the direct limit of the semi-cosimplicial category Δ_S . This becomes evident when considering the following representation of S^+ which in particular motivates to address S^+ as the *partial shifts monoid* and its generators h_k as *partial shifts*.

It is also worthwhile to remark that while S^+ is a quotient of the monoid F^+ , it is *not* a sub-monoid of F^+ . In particular, F^+ is cancellative (see [22]), whereas S^+ is not. For instance $h_k h_k = h_{k+1} h_k$ does not imply that $h_k = h_{k+1}$.

Lemma 2.2.1 (Partial shifts). *The maps $(\theta_k)_{k \geq 0}: \mathbb{N}_0 \rightarrow \mathbb{N}_0$, defined by*

$$\theta_k(n) = \begin{cases} n+1 & \text{if } n \geq k, \\ n & \text{if } n < k, \end{cases}$$

satisfy the relations $\theta_k \theta_\ell = \theta_{\ell+1} \theta_k$ for $0 \leq k \leq \ell < \infty$.

2.3. Noncommutative probability spaces and Markov maps. Throughout, a (*noncommutative*) *probability space* (\mathcal{M}, ψ) consists of a von Neumann algebra \mathcal{M} and a faithful normal state ψ on \mathcal{M} . The identity of \mathcal{M} will be denoted by $\mathbb{1}_{\mathcal{M}}$, or simply by $\mathbb{1}$ when the context is clear. Throughout, $\bigvee_{i \in I} \mathcal{M}_i$ denotes the von Neumann algebra generated by the family of von Neumann algebras $\{\mathcal{M}_i\}_{i \in I} \subset \mathcal{M}$ for $I \subset \mathbb{N}_0$. If \mathcal{M} is abelian and acts on a separable Hilbert space, then (\mathcal{M}, ψ) is isomorphic to $(L^\infty(\Omega, \Sigma, \mu), \int_\Omega \cdot d\mu)$ for some standard probability space (Ω, Σ, μ) .

Definition 2.3.1. An *endomorphism* α of a probability space (\mathcal{M}, ψ) is a $*$ -homomorphism on \mathcal{M} satisfying the following additional properties:

- (i) $\alpha(\mathbb{1}_{\mathcal{M}}) = \mathbb{1}_{\mathcal{M}}$ (unitality);
- (ii) $\psi \circ \alpha = \psi$ (stationarity);
- (iii) α and the modular automorphism group σ_t^ψ commute for all $t \in \mathbb{R}$ (modularity).

The set of endomorphisms of (\mathcal{M}, ψ) is denoted by $\text{End}(\mathcal{M}, \psi)$. We note that an endomorphism of (\mathcal{M}, ψ) is automatically injective. Similarly, $\text{Aut}(\mathcal{M}, \psi)$ denotes the automorphisms of (\mathcal{M}, ψ) .

Definition 2.3.2. Let (\mathcal{M}, ψ) and (\mathcal{N}, φ) be two noncommutative probability spaces. A linear map $T: \mathcal{M} \rightarrow \mathcal{N}$ is called a (ψ, φ) -*Markov map* if the following conditions are satisfied:

- (i) T is completely positive;
- (ii) T is unital;
- (iii) $\varphi \circ T = \psi$;
- (iv) $T \circ \sigma_t^\psi = \sigma_t^\varphi \circ T$, for all $t \in \mathbb{R}$.

Here σ^ψ and σ^φ denote the modular automorphism groups of (\mathcal{M}, ψ) and (\mathcal{N}, φ) , respectively. If $(\mathcal{M}, \psi) = (\mathcal{N}, \varphi)$, we say that T is a ψ -Markov map on \mathcal{M} . Conditions (i) to (iii) imply that a Markov map is automatically normal. The condition (iv) is equivalent to the condition that a unique Markov map $T^*: (\mathcal{N}, \varphi) \rightarrow (\mathcal{M}, \psi)$ exists such that

$$\psi(T^*(y)x) = \varphi(yT(x)) \quad (x \in \mathcal{M}, y \in \mathcal{N}).$$

The Markov map T^* is called the *adjoint* of T and T is called *self-adjoint* if $T = T^*$. We note that condition (iv) is automatically satisfied whenever ψ and φ are tracial, in particular for abelian von Neumann algebras \mathcal{M} and \mathcal{N} .

Remark 2.3.3. More generally in some literature, a Markov map T on a von Neumann algebra \mathcal{M} is understood to be a unital normal completely positive linear map from \mathcal{M} to itself. Thus the above notion of a ψ -Markov map T on \mathcal{M} is more restrictive, as the existence of a *faithful* normal state ψ with the stationarity condition $\psi \circ T = \psi$ and a modular condition are stipulated (as also done in [44, Definition 1.1]). In particular, as recurrence for Markov maps in noncommutative probability is defined via support properties of stationary normal states (see [38]), every non-zero orthogonal projection $p \in \mathcal{M}$ is positive recurrent with respect to a Markov map T on (\mathcal{M}, ψ) , as the faithful state ψ has the support projection $\mathbb{1}_{\mathcal{M}} \in \mathcal{M}$.

We recall for the convenience of the reader the definition of conditional expectations in the present framework of noncommutative probability spaces.

Definition 2.3.4. Let (\mathcal{M}, ψ) be a noncommutative probability space, and \mathcal{N} be a von Neumann subalgebra of \mathcal{M} . A linear map $E: \mathcal{M} \rightarrow \mathcal{N}$ is called a *conditional expectation* if it satisfies the following conditions:

- (i) $E(x) = x$ for all $x \in \mathcal{N}$;
- (ii) $\|E(x)\| \leq \|x\|$ for all $x \in \mathcal{M}$;
- (iii) $\psi \circ E = \psi$.

Such a conditional expectation exists if and only if \mathcal{N} is globally invariant under the modular automorphism group of (\mathcal{M}, ψ) (see [69], [70] and [71]). The von Neumann subalgebra \mathcal{N} is called ψ -conditioned if this condition is satisfied. Note that such a conditional expectation is automatically normal and uniquely determined by ψ . In particular, a conditional expectation is a Markov map and satisfies the module property $E(axb) = aE(x)b$ for $a, b \in \mathcal{N}$ and $x \in \mathcal{M}$.

2.4. Noncommutative random variables and distributional invariance principles. Let (\mathcal{A}, φ) and (\mathcal{M}, ψ) be two probability spaces. A (*noncommutative*) *random variable* ι_0 is an injective *-homomorphism $\iota_0: \mathcal{A} \rightarrow \mathcal{M}$ satisfying two additional properties:

- (i) $\varphi = \psi \circ \iota_0$;
- (ii) $\iota_0(\mathcal{A})$ is ψ -conditioned.

A random variable will also be addressed as the mapping $\iota_0: (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$. If $\tilde{\iota}_0: (\mathcal{A}, \varphi) \rightarrow (\tilde{\mathcal{M}}, \tilde{\psi})$ is another random variable, then ι_0 and $\tilde{\iota}_0$ have the same moment sequence and thus are identically distributed. Given the (identically distributed)

sequence of random variables

$$\iota \equiv (\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi),$$

the family $\mathcal{A}_\bullet \equiv \{\mathcal{A}_I\}_{I \subset \mathbb{N}_0}$, with von Neumann subalgebras $\mathcal{A}_I := \bigvee_{i \in I} \iota_i(\mathcal{A})$, is called the *canonical local filtration (generated by ι)*. The sequence ι is said to be *minimal* if $\mathcal{A}_{\mathbb{N}_0} = \mathcal{M}$. A sequence ι can always be turned into a minimal sequence by restriction. If

$$\tilde{\iota} \equiv (\tilde{\iota}_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\tilde{\mathcal{M}}, \tilde{\psi})$$

is another sequence of random variables, then ι and $\tilde{\iota}$ are said to have the *same distribution*, in symbols $(\iota_0, \iota_1, \iota_2, \dots) \stackrel{\text{distr}}{=} (\tilde{\iota}_0, \tilde{\iota}_1, \tilde{\iota}_2, \dots)$ or just $\iota \stackrel{\text{distr}}{=} \tilde{\iota}$, if

$$\psi(\iota_{k_1}(a_1)\iota_{k_2}(a_2)\cdots\iota_{k_n}(a_n)) = \tilde{\psi}(\tilde{\iota}_{k_1}(a_1)\tilde{\iota}_{k_2}(a_2)\cdots\tilde{\iota}_{k_n}(a_n))$$

for all $k_1, k_2, \dots, k_n \in \mathbb{N}_0$, $a_1, a_2, \dots, a_n \in \mathcal{A}$ and $n \in \mathbb{N}$.

Definition 2.4.1. The sequence of random variables $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ is said to be

- (i) *stationary* if $(\iota_0, \iota_1, \iota_2, \dots) \stackrel{\text{distr}}{=} (\iota_n, \iota_{n+1}, \iota_{n+2}, \dots)$ for all $n \in \mathbb{N}$;
- (ii) *spreadable* if $(\iota_0, \iota_1, \iota_2, \dots) \stackrel{\text{distr}}{=} (\iota_{n_0}, \iota_{n_1}, \iota_{n_2}, \dots)$ for any increasing subsequence (n_0, n_1, n_2, \dots) of $(0, 1, 2, \dots)$;
- (iii) *exchangeable* if $(\iota_0, \iota_1, \iota_2, \dots) \stackrel{\text{distr}}{=} (\iota_{\sigma(0)}, \iota_{\sigma(1)}, \iota_{\sigma(2)}, \dots)$ for all permutations $\sigma \in \mathbb{S}_\infty$.

Here \mathbb{S}_∞ denotes the group of all finite permutations on the set \mathbb{N}_0 such that the Coxeter generator $\sigma_k \in \mathbb{S}_\infty$ is the transposition $(k-1, k)$.

It is elementary to verify that one has the following hierarchy of invariance principles:

$$\text{exchangeability} \implies \text{spreadability} \implies \text{stationarity}.$$

These three distributional invariance principles can be equivalently reformulated.

Proposition 2.4.2. *Suppose $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ is a minimal sequence of random variables.*

- (i) *The sequence ι is stationary if and only if there exists $\alpha \in \text{End}(\mathcal{M}, \psi)$ such that $\iota_n = \alpha^n \iota_0$ for all $n \in \mathbb{N}$.*
- (ii) *The sequence ι is spreadable if and only if there exists a representation $\varrho: S^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ such that $\iota_0 = \varrho(h_k)\iota_0$ for all $k \geq 1$ and $\iota_n = \varrho(h_0^n)\iota_0$ for all $n \in \mathbb{N}$.*
- (iii) *The sequence ι is exchangeable if and only if there exists a representation $\rho_{\text{perm}}: \mathbb{S}_\infty \rightarrow \text{Aut}(\mathcal{M}, \psi)$ such that $\iota_0 = \rho_{\text{perm}}(\sigma_k)\iota_0$ for $k \geq 1$ and $\iota_n = \rho_{\text{perm}}(\sigma_n \sigma_{n-1} \cdots \sigma_1)\iota_0$ for all $n \in \mathbb{N}$.*

Proof. For (i) see [53]. For (ii) see [53, 31]. For (iii) see [40]. \square

The equivalent formulation of spreadability in (ii) and the simple observation that S^+ is a quotient of the Thompson monoid F^+ catalyzed our introduction of partial spreadability in Definition 1.0.3 as a novel distributional invariance principle. This implies the extended hierarchy:

$$\text{exchangeability} \implies \text{spreadability} \implies \text{partial spreadability}$$

\implies stationarity.

2.5. Noncommutative independence and Markovianity. Out of Kümmerer’s investigations on the structure of noncommutative Markov dilations (see for example, [56] and [57]), it emerged that Popa’s geometric notion of commuting squares provides a rich framework for noncommutative independence (see [64]). This notion also manifests itself in the noncommutative extended de Finetti theorem, Theorem 1.0.4. After having introduced commuting squares of von Neumann algebras and some of their properties, as they are well-known in subfactor theory, we reinterpret these geometric objects from the viewpoint of noncommutative probability theory, to define (conditional) commuting square (CS) independence. More generally, we use commuting square structures to introduce in Definition 2.5.10 the notion of a ‘local Markov filtration’ in the noncommutative setting and relate it to more traditional notions of Markovianity for (noncommutative) stochastic processes. Our approach is motivated by Kümmerer’s notion of a Markov dilation (see [56, Subsection 2.2] or for example [44, Section 4]) and furthermore supported by our investigations on distributional invariance principles emerging from the Thompson monoid F^+ .

We recall some equivalent properties as they serve to define commuting squares in subfactor theory (see for example [43, 49]) and as they are familiar from conditional independence in classical probability.

Proposition 2.5.1. *Let $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2$ be ψ -conditioned von Neumann subalgebras of the probability space (\mathcal{M}, ψ) such that $\mathcal{M}_0 \subset (\mathcal{M}_1 \cap \mathcal{M}_2)$. Then the following are equivalent:*

- (i) $E_{\mathcal{M}_0}(xy) = E_{\mathcal{M}_0}(x)E_{\mathcal{M}_0}(y)$ for all $x \in \mathcal{M}_1$ and $y \in \mathcal{M}_2$;
- (ii) $E_{\mathcal{M}_1}E_{\mathcal{M}_2} = E_{\mathcal{M}_0}$;
- (iii) $E_{\mathcal{M}_1}(\mathcal{M}_2) = \mathcal{M}_0$;
- (iv) $E_{\mathcal{M}_1}E_{\mathcal{M}_2} = E_{\mathcal{M}_2}E_{\mathcal{M}_1}$ and $\mathcal{M}_1 \cap \mathcal{M}_2 = \mathcal{M}_0$.

In particular, it holds that $\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2$ if one and thus all of these four assertions are satisfied.

Proof. The tracial case for ψ is proved in [43, Prop. 4.2.1.]. The non-tracial case follows from this, after some minor modifications of the arguments therein. \square

Definition 2.5.2. The inclusions

$$\begin{array}{ccc} \mathcal{M}_2 & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{M}_0 & \subset & \mathcal{M}_1 \end{array}$$

as given in Proposition 2.5.1 are said to form a *commuting square (of von Neumann algebras)* if one (and thus all) of the equivalent conditions (i) to (iv) are satisfied in Proposition 2.5.1.

Notation 2.5.3. We write $I < J$ for two subsets $I, J \subset \mathbb{N}_0$ if $i < j$ for all $i \in I$ and $j \in J$. The cardinality of I is denoted by $|I|$. For $N \in \mathbb{N}_0$, we denote by $I + N$ the shifted set $\{i + N \mid i \in I\}$. Finally, $\mathcal{I}(\mathbb{N}_0)$ denote set of all ‘intervals’ of \mathbb{N}_0 , i.e. sets of the form $[m, n] := \{m, m + 1, \dots, n\}$ or $[m, \infty) := \{m, m + 1, \dots\}$ for $0 \leq m \leq n < \infty$.

Definition 2.5.4. Let (\mathcal{M}, ψ) be a probability space with three ψ -conditioned von Neumann subalgebras \mathcal{M}_0 , \mathcal{M}_1 and \mathcal{M}_2 . Then \mathcal{M}_1 and \mathcal{M}_2 are said to be *CS independent over \mathcal{M}_0* or *conditionally CS independent* if the inclusions

$$\begin{array}{ccc} \mathcal{M}_2 \vee \mathcal{M}_0 & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{M}_0 & \subset & \mathcal{M}_1 \vee \mathcal{M}_0 \end{array}$$

form a commuting square.

The inclusion $\mathcal{M}_0 \subset (\mathcal{M}_1 \cap \mathcal{M}_2)$ is *not* assumed in this definition, and its failure occurs frequently in the context of distributional invariance principles, see for example [53, Example 4.6].

Definition 2.5.5. Let \mathcal{N} be a von Neumann subalgebra of (\mathcal{M}, ψ) . A family of von Neumann subalgebras $\{\mathcal{A}_n\}_{n \in \mathbb{N}_0}$ of (\mathcal{M}, ψ) is called

- (i) *order CS independent over \mathcal{N}* if $\bigvee_{i \in I} \mathcal{A}_i$ and $\bigvee_{j \in J} \mathcal{A}_j$ are CS independent over \mathcal{N} for any $I, J \subset \mathbb{N}_0$ with $I < J$ or $J < I$;
- (ii) *full CS independent over \mathcal{N}* if \mathcal{A}_I and \mathcal{A}_J are CS independent over \mathcal{N} for any $I, J \subset \mathbb{N}_0$ with $I \cap J = \emptyset$.

The sequence of random variables $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ with canonical local filtration $\{\mathcal{A}_I\}_{I \subset \mathbb{N}_0}$ is called

- (i') *order CS independent over \mathcal{N}* if \mathcal{A}_I and \mathcal{A}_J are CS independent over \mathcal{N} for any $I, J \subset \mathbb{N}_0$ with $I < J$ or $J < I$;
- (ii') *full CS independent over \mathcal{N}* if \mathcal{A}_I and \mathcal{A}_J are CS independent over \mathcal{N} for any $I, J \subset \mathbb{N}_0$ with $I \cap J = \emptyset$.

Clearly, full CS independence over \mathcal{N} implies order CS independence over \mathcal{N} . Occasionally both of them will be just addressed as conditional CS independence. The noncommutative extended de Finetti theorem, Theorem 1.0.4, establishes that a spreadable sequence ι is full CS independence over the tail algebra $\mathcal{N} = \bigcap_{n \geq 0} \bigvee_{k \geq n} \iota_k(\mathcal{A})$.

We address next the basic notions of Markovianity in noncommutative probability. Commonly, Markovianity is understood as a property of random variables relative to a filtration of the underlying probability space. Our investigations from the viewpoint of distributional invariance principles reveal that the phenomenon of ‘Markovianity’ emerges without reference to any stochastic process already on the level of a family of von Neumann subalgebras, indexed by the partially ordered set of all ‘intervals’ $\mathcal{I}(\mathbb{N}_0)$. As commonly the index set of a filtration is understood to be totally ordered [73] and guided by related notions for random Markov fields or generalized stochastic processes, we refer to such partially indexed families as ‘local filtrations’. This is motivated by the fact that sequences of random variables are simple examples of one-dimensional random fields.

Definition 2.5.6. A family of ψ -conditioned von Neumann subalgebras $\mathcal{M}_\bullet \equiv \{\mathcal{M}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ of the probability space (\mathcal{M}, ψ) is called a *local filtration (of (\mathcal{M}, ψ))* if

$$I \subset J \implies \mathcal{M}_I \subset \mathcal{M}_J. \quad (\text{Isotony})$$

A local filtration \mathcal{M}_\bullet is said to be *locally minimal* if $\mathcal{M}_I \vee \mathcal{M}_J = \mathcal{M}_K$ whenever $I, J, K \in \mathcal{I}(\mathbb{N}_0)$ with $I \cup J = K$.

The isotony property ensures that inclusions are valid as they are assumed for commuting squares. To be more precise, it holds that

$$\begin{array}{ccc} \mathcal{M}_I & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{M}_K & \subset & \mathcal{M}_J \end{array}$$

for $I, J, K \in \mathcal{I}(\mathbb{N}_0)$ with $K \subset (I \cap J)$. Finally, let $\mathcal{N}_\bullet \equiv \{\mathcal{N}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ be another local filtration of (\mathcal{M}, ψ) . Then \mathcal{N}_\bullet is said to be *coarser* than \mathcal{M}_\bullet if $\mathcal{N}_I \subset \mathcal{M}_I$ for all $I \in \mathcal{I}(\mathbb{N}_0)$ and we denote this by $\mathcal{N}_\bullet \prec \mathcal{M}_\bullet$. Occasionally we will address \mathcal{N}_\bullet also as a *local subfiltration* of \mathcal{M}_\bullet .

Definition 2.5.7. Let $\mathcal{M}_\bullet \equiv \{\mathcal{M}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ be a local filtration of (\mathcal{M}, ψ) .

- (i) \mathcal{M}_\bullet is said to be *Markovian* (or a *local Markov filtration*) if $\mathcal{M}_{[0, n-1]}$ and $\mathcal{M}_{[n+1, \infty]}$ are CS independent over $\mathcal{M}_{[n, n]}$ for all $n \geq 1$, i.e. the inclusions

$$\begin{array}{ccc} \mathcal{M}_{[0, n-1]} \vee \mathcal{M}_{[n, n]} & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{M}_{[n, n]} & \subset & \mathcal{M}_{[n+1, \infty]} \vee \mathcal{M}_{[n, n]} \end{array}$$

form a commuting square for any $n \geq 1$.

- (ii) \mathcal{M}_\bullet is said to be *saturated Markovian* (or a *saturated local Markov filtration*), if $\mathcal{M}_{[0, n]}$ and $\mathcal{M}_{[n, \infty]}$ are CS independent over $\mathcal{M}_{[n, n]}$ for all $n \geq 0$, i.e. the inclusions

$$\begin{array}{ccc} \mathcal{M}_{[0, n]} & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{M}_{[n, n]} & \subset & \mathcal{M}_{[n, \infty]} \end{array}$$

form a commuting square for each $n \in \mathbb{N}_0$.

Cast as commuting squares, (saturated) Markovianity of the local filtration \mathcal{M}_\bullet has many equivalent formulations, see Proposition 2.5.1. In particular, corresponding to (i) of Definition 2.5.7,

$$E_{\mathcal{M}_{[0, n-1]} \vee \mathcal{M}_{[n, n]}} E_{\mathcal{M}_{[n, n]} \vee \mathcal{M}_{[n+1, \infty]}} = E_{\mathcal{M}_{[n, n]}} \quad \text{for all } n \geq 1. \quad (\text{M})$$

In the formulation (ii) of Definition 2.5.7, it holds that

$$E_{\mathcal{M}_{[0, n]}} E_{\mathcal{M}_{[n, \infty]}} = E_{\mathcal{M}_{[n, n]}} \quad \text{for all } n \geq 0. \quad (\text{M}')$$

Here $E_{\mathcal{M}_I}$ denotes the ψ -preserving normal conditional expectation from \mathcal{M} onto \mathcal{M}_I .

Lemma 2.5.8. *A saturated local Markov filtration \mathcal{M}_\bullet is a Markovian. A locally minimal local Markov filtration \mathcal{M}_\bullet is saturated.*

In other words, if \mathcal{M}_\bullet is locally minimal, then \mathcal{M}_\bullet has the Markov property (M) if and only if it satisfies (M').

Proof. The isotony property of local filtrations ensures the inclusions $\mathcal{M}_{[0,n-1]} \vee \mathcal{M}_{[n,n]} \subset \mathcal{M}_{[0,n]}$ and $\mathcal{M}_{[n+1,\infty]} \vee \mathcal{M}_{[n,n]} \subset \mathcal{M}_{[n,\infty]}$. Thus

$$\begin{aligned} E_{\mathcal{M}_{[0,n-1]} \vee \mathcal{M}_{[n,n]}} E_{\mathcal{M}_{[n,n]} \vee \mathcal{M}_{[n+1,\infty]}} \\ &= E_{\mathcal{M}_{[0,n-1]} \vee \mathcal{M}_{[n,n]}} E_{\mathcal{M}_{[0,n]}} E_{\mathcal{M}_{[n,\infty]}} E_{\mathcal{M}_{[n,n]} \vee \mathcal{M}_{[n+1,\infty]}} \\ &= E_{\mathcal{M}_{[0,n-1]} \vee \mathcal{M}_{[n,n]}} E_{\mathcal{M}_{[n,n]}} E_{\mathcal{M}_{[n,n]} \vee \mathcal{M}_{[n+1,\infty]}} \\ &= E_{\mathcal{M}_{[n,n]}}. \end{aligned}$$

Consequently (M') implies (M). Local minimality of \mathcal{M}_\bullet ensures $\mathcal{M}_{[0,n-1]} \vee \mathcal{M}_{[n,n]} = \mathcal{M}_{[0,n]}$ and $\mathcal{M}_{[n+1,\infty]} \vee \mathcal{M}_{[n,n]} = \mathcal{M}_{[n,\infty]}$. That the property (M) implies (M') is immediate under this minimality assertion. \square

In the following we will not distinguish explicitly between the properties (M) and (M') for local Markov filtrations. Also we will address 'saturated Markovianity' just as 'Markovianity', as our main results aim anyway at establishing the apparently stronger property (M') and, moreover, both these properties are equivalent for locally minimal local filtrations. In particular, local Markov filtrations arising from noncommutative stationary Markov processes as in Subsection 2.6 are indeed locally minimal.

Remark 2.5.9. Markovianity of the local filtration \mathcal{M}_\bullet , as introduced in Definition 2.5.7, may not transfer to a local subfiltration \mathcal{N}_\bullet . To be more precise, the inclusions

$$\begin{array}{ccc} \mathcal{N}_{[0,n]} & \subset & \mathcal{N} \\ \cup & & \cup \\ \mathcal{N}_{[n,n]} & \subset & \mathcal{N}_{[n,\infty]} \end{array}$$

may not form a commuting square, even though the corresponding inclusions of the local filtration \mathcal{M}_\bullet do. In classical probability, this observation is connected to the fact that a function of a Markov process may not be Markovian. We address this phenomenon more in detail in Subsection 3.3 from the viewpoint of noncommutative probability.

Definition 2.5.10. Let $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ be a sequence of random variables with canonical local filtration $\mathcal{A}_\bullet \equiv \{\mathcal{A}_I := \bigvee_{n \in I} \iota_n(\mathcal{A})\}_{I \in \mathcal{I}(\mathbb{N}_0)}$. Furthermore let $\mathcal{M}_\bullet \equiv \{\mathcal{M}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ be another local filtration of (\mathcal{M}, ψ) .

- (i) ι is said to be *adapted to (the local filtration) \mathcal{M}_\bullet* if $\mathcal{A}_I \subset \mathcal{M}_I$ for all $I \in \mathcal{I}(\mathbb{N}_0)$.
- (ii) ι is said to be *\mathcal{M}_\bullet -Markovian* (or an *\mathcal{M}_\bullet -Markov sequence*) if ι is adapted to \mathcal{M}_\bullet and

$$E_{\mathcal{M}_{[0,n]}} \iota_{n+1} = E_{\mathcal{A}_{[n,n]}} \iota_{n+1} \quad (n \geq 0).$$

An \mathcal{M}_\bullet -Markov sequence ι is just said to be *Markovian* (or a *Markov sequence*) if $\mathcal{M}_\bullet = \mathcal{A}_\bullet$.

It is elementary to verify that an \mathcal{M}_\bullet -Markovian sequence ι is \mathcal{N}_\bullet -Markovian whenever $\mathcal{A}_\bullet \prec \mathcal{N}_\bullet \prec \mathcal{M}_\bullet$. In particular, any \mathcal{M}_\bullet -Markov sequence is Markovian. We emphasize that there exist non-Markovian sequences ι which are adapted to a local Markov filtration \mathcal{M}_\bullet . 'Trivial' examples for such sequences are provided in Remark 2.5.12, and 'less trivial' classical examples are the topic of Subsection 3.3. Let us remark here without further ado, that such non-Markovian sequences are especially of relevance

for distributional invariance principles, as the latter are often about the characterization of so-called ‘mixtures of stochastic processes’.

The next result states a sufficient condition ensuring that the Markovianity of a local filtration is inherited by a sequence.

Lemma 2.5.11. *Suppose the sequence ι is adapted to the local Markov filtration \mathcal{M}_\bullet such that the inclusions*

$$\begin{array}{ccc} \mathcal{A}_{[0,\infty]} & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{A}_{[n,n]} & \subset & \mathcal{M}_{[n,n]} \end{array}$$

form a commuting square for all $n \geq 0$. Then ι is $(\mathcal{M}_\bullet\text{-})$ Markovian.

In particular, if the canonical (local) filtration of ι is Markovian as in Definition 2.5.7, then ι is a Markov sequence as in Definition 2.5.10 (ii).

Proof. This is immediate from the equations

$$\begin{aligned} E_{\mathcal{M}_{[0,n]}} E_{\mathcal{A}_{[n+1,n+1]}} &= E_{\mathcal{M}_{[0,n]}} E_{\mathcal{M}_{[n,\infty]}} E_{\mathcal{A}_{[n+1,n+1]}} \\ &= E_{\mathcal{M}_{[n,n]}} E_{\mathcal{A}_{[n+1,n+1]}} = E_{\mathcal{M}_{[n,n]}} E_{\mathcal{A}_{[0,\infty]}} E_{\mathcal{A}_{[n+1,n+1]}} \\ &= E_{\mathcal{A}_{[n,n]}} E_{\mathcal{A}_{[n+1,n+1]}}. \end{aligned}$$

□

Remark 2.5.12. The Markovianity of a local filtration in Definition 2.5.10 (i) should be understood with some care, as it permits ‘trivial’ filtrations. For example, given the probability space (\mathcal{M}, ψ) , the local filtration $\mathcal{M}_\bullet \equiv \{\mathcal{M}_I = \mathcal{M}\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ is Markovian and any sequence of random variables is adapted to it. Note also that a constant sequence ι (with $\iota_n = \iota_0$ for all $n \geq 0$) is adapted to a local Markov filtration \mathcal{M}_\bullet whenever $\iota_0(\mathcal{A}) \subset \mathcal{M}_{[n,n]}$ for all $n \geq 0$.

Remark 2.5.13. Presently we do not know in the full generality of our noncommutative framework if the canonical local filtration of a Markov sequence is Markovian (in the sense of Definition 2.5.7). But this is true under additional algebraic conditions on the Markov sequence which fully include the classical case, see Subsection 3.2.

2.6. Noncommutative stationary processes and dilations. We introduce unilateral noncommutative stationary processes, as they underly the approach to distributional invariance principles in [53, 40]. Furthermore we present Anantharaman-Delaroche’s notion of factorizable Markov maps [5] and unilateral versions of dilations of Markov maps as they are subject of Kümmerer’s approach to noncommutative stationary Markov processes [56]. The existence of such dilations is actually equivalent to the factorizability of Markov maps [44].

Definition 2.6.1. A *(unilateral) stationary process* $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$ consists of a probability space (\mathcal{M}, ψ) , a ψ -conditioned subalgebra $\mathcal{A}_0 \subset \mathcal{M}$, and an endomorphism $\alpha \in \text{End}(\mathcal{M}, \psi)$. The sequence

$$(\iota_n)_{n \geq 0} : (\mathcal{A}_0, \psi_0) \rightarrow (\mathcal{M}, \psi), \quad \iota_n := \alpha^n|_{\mathcal{A}_0} = \alpha^n \iota_0,$$

is called the *sequence of random variables associated to* $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$. Here ψ_0 denotes the restriction of ψ from \mathcal{M} to \mathcal{A}_0 and ι_0 denotes the inclusion map of \mathcal{A}_0 in \mathcal{M} .

Definition 2.6.2. A stationary process $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$ is said to have property ‘A’ if its associated sequence of random variables $(\iota_n)_{n \geq 0}: (\mathcal{A}_0, \psi_0) \rightarrow (\mathcal{M}, \psi)$ has property ‘A’. For example, $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$ is minimal if $(\iota_n)_{n \geq 0}$ is minimal.

Definition 2.6.3. The stationary process $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$ is called a (*unilateral noncommutative*) *stationary Markov process* if its canonical local filtration

$$\left\{ \mathcal{A}_I := \bigvee_{i \in I} \alpha^i \iota_0(\mathcal{A}_0) \right\}_{I \in \mathcal{I}(\mathbb{N}_0)}$$

is Markovian. If this process is minimal, then the endomorphism α is also called a *Markov shift* with generator \mathcal{A}_0 .

The associated ψ_0 -Markov map $T = \iota_0^* \alpha \iota_0$, where ι_0 is the inclusion map of \mathcal{A}_0 in \mathcal{M} and ψ_0 the restriction of ψ to \mathcal{A}_0 , is often called the *transition operator* of the given Markov process.

Anantharaman-Delaroche has introduced in [5, Definition 6.2] the notion of factorizable maps which we recall here in the setting relevant to our approach.

Definition 2.6.4 ([5]). Let (\mathcal{A}, φ) be a probability space. A φ -Markov map T on \mathcal{A} is called *factorizable* if there exists a probability space (\mathcal{M}, ψ) and random variables $\iota_1, \iota_2: (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ such that $T = \iota_2^* \iota_1$.

The factorizability of a Markov map is actually equivalent to the existence of certain dilations of a Markov map as they are studied by Kümmerer in [56]. Here we are interested in the following straightforward unilateral modifications of the original bilateral notions of dilation and Markov dilation in [56].

Definition 2.6.5 ([56]). Let (\mathcal{A}, φ) be a probability space. A φ -Markov map T on \mathcal{A} is said to admit a (*unilateral state-preserving*) *dilation* if there exists a probability space (\mathcal{M}, ψ) , an endomorphism $\alpha \in \text{End}(\mathcal{M}, \psi)$ and a (φ, ψ) -Markov map $\iota_0: \mathcal{A} \rightarrow \mathcal{M}$ such that, for all $n \in \mathbb{N}_0$,

$$T^n = \iota_0^* \alpha^n \iota_0.$$

Such a dilation of T is denoted by the quadruple $(\mathcal{M}, \psi, \alpha, \iota_0)$ and is said to be *minimal* if $\mathcal{M} = \bigvee_{n \in \mathbb{N}_0} \alpha^n \iota_0(\mathcal{A})$. The quadruple $(\mathcal{M}, \psi, \alpha, \iota_0)$ is called a *dilation of first order* if the equality $T = \iota_0^* \alpha \iota_0$ alone holds.

Actually it follows from the case $n = 0$ that the (φ, ψ) -Markov map ι_0 is a random variable from (\mathcal{A}, φ) to (\mathcal{M}, ψ) such that $\iota_0 \iota_0^*$ is the ψ -preserving conditional expectation from \mathcal{M} onto $\iota_0(\mathcal{A})$.

Definition 2.6.6 ([56]). The dilation $(\mathcal{M}, \psi, \alpha, \iota_0)$ of the φ -Markov map T on \mathcal{A} (as introduced in Definition 2.6.5) is said to be a (*unilateral state-preserving*) *Markov dilation* if the local filtration $\left\{ \mathcal{A}_I := \bigvee_{n \in I} \alpha^n \iota_0(\mathcal{A}) \right\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ is Markovian.

Remark 2.6.7. A dilation of a φ -Markov map T on \mathcal{A} may not be a Markov dilation. This is discussed in [60, Section 3] where it is shown that Varilly has constructed a dilation in [72] which is not a Markov dilation. We are grateful to B. Kümmerer for bringing this to our attention [59].

Definition 2.6.8. Let (\mathcal{A}, φ) be a probability space and T be a φ -Markov map on \mathcal{A} . A dilation (of first order) $(\mathcal{M}, \psi, \alpha, \iota_0)$ of T is called a *tensor dilation* if the conditional expectation $\iota_0 \iota_0^*: \mathcal{M} \rightarrow \iota_0(\mathcal{A})$ is of tensor type, that is, there exists a von Neumann subalgebra \mathcal{C} of \mathcal{M} with faithful normal state χ such that $\mathcal{M} = \mathcal{A} \otimes \mathcal{C}$ and $(\iota_0 \iota_0^*)(a \otimes x) = \chi(x)(a \otimes \mathbb{1}_{\mathcal{C}})$ for all $a \in \mathcal{A}$, $x \in \mathcal{C}$.

The following result was obtained by Haagerup and Musat in [44]. We reformulate it here, using unilateral instead of bilateral notions of dilations.

Theorem 2.6.9 ([44]). *Let T be a φ -Markov map on the von Neumann algebra \mathcal{A} equipped with a faithful normal state φ . Then the following are equivalent:*

- (a) T is factorizable (in the sense of Definition 2.6.4).
- (b) T admits a dilation (in the sense of Definition 2.6.5).
- (c) T admits a Markov dilation (in the sense of Definition 2.6.6).

We refer the reader to the proof given in [44, Theorem 4.4], as its relevant arguments transfer to our present setting. Note that clearly (c) implies (b). Furthermore (b) implies (a) by putting $\iota_1 := \alpha \iota$ and $\iota_2 := \iota$ such that $T = \iota_2^* \iota_1$. So the main task is to prove that (a) implies (c), which makes use of [5, Theorem 6.6].

Let us next relate the above unilateral notions of dilations and stationary processes. It is immediate that a dilation $(\mathcal{M}, \psi, \alpha, \iota_0)$ of the φ -Markov map T on \mathcal{A} gives rise to the stationary process $(\mathcal{M}, \psi, \alpha, \iota_0(\mathcal{A}))$. Furthermore this stationary process is Markovian if and only if the dilation is a Markov dilation, as evident from the definitions. Conversely, a stationary Markov process yields a dilation (and thus a Markov dilation) as it was shown by Kümmerer in the setting of bilateral stationary Markov processes (see [56, Proposition 2.2.7]). We adapt next its proof to the unilateral setting, as we will need this result in Theorem 5.1.6.

Proposition 2.6.10. *Let $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$ be a stationary Markov process and $T = \iota_0^* \alpha \iota_0$ be the corresponding transition operator where ι_0 is the inclusion map of \mathcal{A}_0 into \mathcal{M} . Then $(\mathcal{M}, \psi, \alpha, \iota_0)$ is a dilation of T . In other words, the following diagram commutes for all $n \in \mathbb{N}_0$:*

$$\begin{array}{ccc} (\mathcal{A}_0, \psi_0) & \xrightarrow{T^n} & (\mathcal{A}_0, \psi_0) \\ \downarrow \iota_0 & & \uparrow \iota_0^* \\ (\mathcal{M}, \psi) & \xrightarrow{\alpha^n} & (\mathcal{M}, \psi) \end{array} \quad .$$

Here ψ_0 denotes the restriction of ψ to \mathcal{A}_0 .

Proof. For $I \subset \mathbb{N}_0$, let $\mathcal{A}_I = \bigvee_{n \in I} \alpha^n(\iota_0(\mathcal{A}_0))$ and let $E_{\mathcal{A}_I}$ be the unique normal ψ -preserving conditional expectation onto \mathcal{A}_I , which we write as P_I for brevity. Then the intertwiner relation $P_{I+k} \circ \alpha^k = \alpha^k \circ P_I$, with $k \in \mathbb{N}_0$ and $I \subset \mathbb{N}_0$, can be seen using the following argument. Since $\mathcal{A}_{I+k} = \alpha^k(\mathcal{A}_I)$, it holds for $y \in \mathcal{A}_I$ and $x \in \mathcal{M}$ that

$$\begin{aligned} \psi(\alpha^k(y) P_{I+k} \alpha^k(x)) &= \psi(\alpha^k(y) \alpha^k(x)) = \psi(\alpha^k(yx)) = \psi(yx) \\ &= \psi(y P_I(x)) = \psi(\alpha^k(y) \alpha^k P_I(x)). \end{aligned}$$

As the functionals $\{\psi(\alpha^k(y) \cdot) \mid y \in \mathcal{A}_I\}$, considered as elements in the predual of \mathcal{A}_{I+k} , are norm dense, we conclude $P_{I+k}\alpha^k(x) = \alpha^k P_I(x)$ for all $x \in \mathcal{M}$. In particular, we get

$$\begin{aligned} P_{[k-1, k-1]}\alpha^k \iota_0 &= P_{[k-1, k-1]}\alpha^{k-1} \alpha \iota_0 \\ &= \alpha^{k-1} P_{[0, 0]}\alpha \iota_0 = \alpha^{k-1} \iota_0 T \quad (k \in \mathbb{N}), \end{aligned} \quad (2.6.1)$$

where we have used $P_{[0, 0]} = \iota_0 \iota_0^*$ for the last equality. Now we prove the dilation property by induction. We know that $\iota_0^* \alpha^n \iota_0 = T^n$ is true for $n = 0, 1$. Suppose $\iota_0^* \alpha^n \iota_0 = T^n$ for some $n \in \mathbb{N}_0$. Then

$$\begin{aligned} \iota_0^* \alpha^{n+1} \iota_0 &= \iota_0^* P_{[0, 0]}\alpha^{n+1} \iota_0 && \text{(as } \iota_0^* \iota_0 = \text{Id and } \iota_0 \iota_0^* = P_{[0, 0]}) \\ &= \iota_0^* P_{[0, 0]} P_{[0, n]}\alpha^{n+1} \iota_0 && \text{(as } \mathcal{A}_{[0, 0]} \subset \mathcal{A}_{[0, n]}) \\ &= \iota_0^* P_{[0, 0]} P_{[n, n]}\alpha^{n+1} \iota_0 && \text{(by Markovianity)} \\ &= \iota_0^* \alpha^n \iota_0 T && \text{(by Equation (2.6.1))} \\ &= T^n T = T^{n+1} && \text{(by induction hypothesis).} \end{aligned}$$

□

The dilation $(\mathcal{M}, \psi, \alpha, \iota_0)$ in Proposition 2.6.10 is of course a Markov dilation, as its local filtration is that of the stationary Markov process.

The following formula for ‘pyramidally time ordered correlations’ is obtained by an application of the Markov property and the intertwiner relation (see for example [57, Subsection 2.2] or [39, Subsection 2.1]).

Lemma 2.6.11. *Let $(\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ be a stationary Markov sequence of random variables with transition operator $T = \iota_0^* \alpha \iota_0$, where $\alpha \in \text{End}(\mathcal{M}, \psi)$ such that $\iota_n = \alpha^n \iota_0$, $n \in \mathbb{N}_0$. Suppose $k_1 < k_2 < \dots < k_n$. Then for all $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{A}$,*

$$\begin{aligned} &\psi(\iota_{k_1}(a_1^*) \cdots \iota_{k_n}(a_n^*) \iota_{k_n}(b_n) \cdots \iota_{k_1}(b_1)) \\ &= \varphi(a_1^* T^{k_2 - k_1}(a_2^* T^{k_3 - k_2}(a_3 \cdots T^{k_n - k_{n-1}}(a_n^* b_n) \cdots) b_2) b_1). \end{aligned}$$

Proof. Let P_I be the the unique normal ψ -preserving conditional expectation onto \mathcal{A}_I , where $\mathcal{A}_I = \bigvee_{n \in I} \alpha^n(\iota_0(\mathcal{A}))$. Then using that $(\mathcal{M}, \psi, \alpha, \iota_0)$ is a dilation of T , we get

$$\begin{aligned} P_{[p, p]}\alpha^q \iota_0 &= P_{[p, p]}\alpha^p \alpha^{q-p} \iota_0 \\ &= \alpha^p P_{[0, 0]}\alpha^{q-p} \iota_0 = \alpha^p \iota_0 T^{q-p} \quad (p < q). \end{aligned} \quad (2.6.2)$$

Now we compute

$$\begin{aligned} &\psi(\iota_{k_1}(a_1^*) \cdots \iota_{k_n}(a_n^*) \iota_{k_n}(b_n) \cdots \iota_{k_1}(b_1)) \\ &= \psi(\alpha^{k_1}(\iota_0(a_1^*)) \cdots \alpha^{k_n}(\iota_0(a_n^*)) \alpha^{k_n}(\iota_0(b_n)) \cdots \alpha^{k_1}(\iota_0(b_1))) \\ &= \psi(\alpha^{k_1}(\iota_0(a_1^*)) \alpha^{k_2 - k_1}(\iota_0(a_2^*)) \cdots \alpha^{k_n - k_1}(\iota_0(a_n^* b_n)) \cdots (\iota_0(b_1))) \\ &= \psi(P_{[0, k_{n-1} - k_1]}(\iota_0(a_1^*) \cdots \alpha^{k_{n-1} - k_1}(\iota_0(a_{n-1}^*)) \quad \times \\ &\quad \alpha^{k_n - k_1}(\iota_0(a_n b_n^*)) \alpha^{k_{n-1} - k_1}(\iota_0(b_{n-1})) \cdots (\iota_0(b_1)))) \end{aligned}$$

$$\begin{aligned}
&= \psi \left(\iota_0(a_1^*) \cdots \alpha^{k_{n-1}-k_1}(\iota_0(a_{n-1}^*)) P_{[0, k_{n-1}-k_1]}(\alpha^{k_n-k_1}(\iota_0(a_n^* b_n))) \right) \times \\
&\quad \alpha^{k_{n-1}-k_1}(\iota_0(b_{n-1})) \cdots \iota_0(b_1) \\
&= \psi \left(\iota_0(a_1^*) \cdots \alpha^{k_{n-1}-k_1}(\iota_0(a_{n-1}^*)) P_{[k_{n-1}-k_1, k_{n-1}-k_1]}(\alpha^{k_n-k_1}(\iota_0(a_n^* b_n))) \right) \times \\
&\quad \alpha^{k_{n-1}-k_1}(\iota_0(b_{n-1})) \cdots \iota_0(b_1) \\
&= \psi \left(\iota_0(a_1^*) \cdots \alpha^{k_{n-1}-k_1}(\iota_0(a_{n-1}^*)) \alpha^{k_{n-1}-k_1} \iota_0 T^{k_n-k_{n-1}}(a_n^* b_n) \right) \times \\
&\quad \alpha^{k_{n-1}-k_1}(\iota_0(b_{n-1})) \cdots \iota_0(b_1) \\
&= \psi \left(\iota_0(a_1^*) \cdots \alpha^{k_{n-1}-k_1} \iota_0(a_{n-1}^* T^{k_n-k_{n-1}}(a_n^* b_n) b_{n-1}) \cdots \iota_0(b_1) \right) \\
&\quad \vdots \\
&= \psi \left(\iota_0(a_1^* T^{k_2-k_1}(a_2^* \cdots T^{k_{n-1}-k_{n-2}}(a_{n-1}^* T^{k_n-k_{n-1}}(a_n^* b_n) b_{n-1}) \cdots b_2) b_1) \right) \\
&= \varphi \left(a_1^* T^{k_2-k_1}(a_2^* \cdots T^{k_{n-1}-k_{n-2}}(a_{n-1}^* T^{k_n-k_{n-1}}(a_n^* b_n) b_{n-1}) \cdots b_2) b_1 \right).
\end{aligned}$$

Here, the fifth line is obtained from the fourth by the module property of conditional expectations, the sixth line is obtained from the fifth by the Markov property, and the seventh from the sixth by (2.6.2) as $k_n - k_1 > k_{n-1} - k_1$. We note that as our random variables are all unital, we can assume without loss of generality that each $k_{l+1} = k_l + 1$ for each $l \in \{1, \dots, n-1\}$. \square

Within our general framework of noncommutative probability, we do not presently know if the canonical local filtration of a stationary Markov sequence is Markovian (in the sense of Definition 2.5.7). However, this is indeed the case if one considers commutative von Neumann algebras or, more generally, von Neumann algebras which are ‘pyramidally generated’ by a sequence of random variables. Let us make this algebraic condition more precise:

Definition 2.6.12. Suppose $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ is a sequence of random variables with canonical local filtration $\mathcal{A}_\bullet \equiv (\mathcal{A}_I)_{I \in \mathcal{I}(\mathbb{N}_0)}$. The von Neumann algebra $\mathcal{A}_{\mathbb{N}_0}$ is said to be *pyramidally generated* by the random variables if elements of the form

$$\iota_n(a_0^*) \iota_{n+1}(a_1^*) \cdots \iota_{n+p-1}(a_{p-1}^*) \iota_{n+p}(a_p^* b_p) \iota_{n+p-1}(b_{p-1}) \cdots \iota_{n+1}(b_1) \iota_n(b_0)$$

are weak*-total in $\mathcal{A}_{[n, n+p]}$ for all $n, p \in \mathbb{N}_0$, where $a_1, \dots, a_p, b_1, \dots, b_p \in \mathcal{A}$.

Proposition 2.6.13. *Suppose $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ is a stationary Markov sequence with canonical local filtration $\mathcal{A}_\bullet \equiv (\mathcal{A}_I)_{I \in \mathcal{I}(\mathbb{N}_0)}$. If $\mathcal{A}_{\mathbb{N}_0}$ is pyramidally generated then \mathcal{A}_\bullet is Markovian.*

If \mathcal{M} (and hence \mathcal{A}) is commutative then $\mathcal{A}_{\mathbb{N}_0}$ is pyramidally generated and thus \mathcal{A}_\bullet is Markovian.

Proof. By (2.6.1) we have that $P_{[k-1, k-1]} \iota_k = \iota_{k-1} T$ for all $k \in \mathbb{N}$. We conclude from this that, for $n, p \in \mathbb{N}_0$ and $x, y \in \mathcal{A}_{[n, n+p]}$, $a \in \mathcal{A}$,

$$\begin{aligned} P_{[0, n]}(x \iota_{n+p+1}(a) y) &= P_{[0, n]} P_{[0, n+p]}(x \iota_{n+p+1}(a) y) \\ &= P_{[0, n]}(x P_{[0, n+p]} \iota_{n+p+1}(a) y) \\ &= P_{[0, n]}(x \iota_{n+p} T(a) y). \end{aligned}$$

A repeated application of the above gives for $a_1, \dots, a_p, b_1, \dots, b_p \in \mathcal{A}$,

$$\begin{aligned} P_{[0, n]}(\iota_n(a_0^*) \cdots \iota_{n+p}(a_p^*) \iota_{n+p+1}(a_{p+1}^* b_{p+1}) \iota_{n+p}(b_p) \cdots \iota_n(b_0)) \\ &= P_{[0, n]}(\iota_n(a_0^* T(a_1^* \cdots T(a_{p+1}^* b_{p+1}) \cdots b_1) b_0)) \\ &= P_{[n, n]}(\iota_n(a_0^* T(a_1^* \cdots T(a_{p+1}^* b_{p+1}) \cdots b_1) b_0)) \\ &= P_{[n, n]}(\iota_n(a_0^*) \cdots \iota_{n+p}(a_p^*) \iota_{n+p+1}(a_{p+1}^* b_{p+1}) \iota_{n+p}(b_p) \cdots \iota_n(b_0)). \end{aligned}$$

So $P_{[0, n]}$ and $P_{[n, n]}$ coincide on a weak*-total subset of $\mathcal{A}_{[n, n+p+1]}$ for every $p \geq 0$. Since $\bigcup_{p \geq 0} \mathcal{A}_{[n, n+p+1]}$ is weak*-dense in $\mathcal{A}_{[n, \infty)}$ it follows $P_{[0, n]} P_{[n, \infty)} = P_{[n, n]}$ by another approximation argument. Thus \mathcal{A}_\bullet is a local Markov filtration. \square

Remark 2.6.14. The assumption of stationarity of the Markov sequence in Proposition 2.6.13 can be dropped so that the result still remains true. To see this, consider a (possibly) inhomogeneous Markov sequence $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ with φ -Markov maps $T_{n+1} := \iota_n^* \iota_{n+1}$ as transition operators (see [39, Subsection 2.1]) on \mathcal{A} which satisfy

$$P_{[n, n]} \iota_{n+1} = \iota_n T_{n+1}$$

for all $n \geq 0$. Here P_I denotes the ψ -preserving conditional expectation onto \mathcal{A}_I as in Proposition 2.6.13. Replacing appropriately the single transition operator T from the stationary setting by the transition operators T_n , all arguments of the proof of Proposition 2.6.13 transfer to the setting of inhomogeneous Markov sequences. Thus the canonical local filtration of a (not necessarily stationary) Markov sequence is Markovian (in the sense of Definition 2.5.7) if $\mathcal{A}_{\mathbb{N}_0}$ is pyramidally-generated.

We close this subsection by providing a noncommutative notion of operator-valued Bernoulli shifts, as we will meet them when constructing representations of the partial shift monoid S^+ in Section 5. The definition of such shifts stems from investigations of Kümmerer on the structure of noncommutative Markov processes in [56], and such shifts can also be seen to emerge from the noncommutative extended de Finetti theorem in [53] (see Theorem 1.0.4).

Definition 2.6.15. The minimal stationary process $(\mathcal{M}, \psi, \beta, \mathcal{B}_0)$ is called a (*unilateral noncommutative full/ordered*) *Bernoulli shift* with generator \mathcal{B}_0 if $\mathcal{M}^\beta \subset \mathcal{B}_0$ and $\{\beta^n(\mathcal{B}_0)\}_{n \in \mathbb{N}_0}$ is full/order CS independent over \mathcal{M}^β .

It is easy to see that a Bernoulli shift $(\mathcal{M}, \psi, \beta, \mathcal{B}_0)$ is a minimal stationary Markov process where the corresponding transition operator $\iota_0^* \beta \iota_0$ is a conditional expectation (onto \mathcal{M}^β , the fixed point algebra of β). Here ι_0 denotes the inclusion map of \mathcal{B}_0 into \mathcal{M} .

3. A NEW DISTRIBUTIONAL INVARIANCE PRINCIPLE IN CLASSICAL PROBABILITY THEORY

Initially the deep connection between Markovianity and certain representations of the Thompson monoid F^+ emerged from investigations on distributional symmetries and invariance principles in noncommutative probability. These pioneering results are of course also available when restricting our framework to that of classical probability. We review in Subsection 3.1 the algebraization procedure of random variables, in particular to make our approach more accessible to readers from traditional probability theory. Continuing in Subsection 3.2, we discuss some of our main definitions and results on ‘partial spreadability’, using their reformulations in terms of properties of classical random variables. Among these reformulated results is the already stated Theorem 1.0.1, a de Finetti theorem for stationary Markov sequences. Subsection 3.3 presents evidence that functions of stationary Markov sequences are partially spreadable. Finally, Subsection 3.4 is about constructing stationary Markov sequences as Markov dilations in classical probability, as featured in Kümmerer’s approach to noncommutative stationary Markov processes. Such an alternative construction turns out to be key for the construction of a representation of the Thompson monoid F^+ , as needed for the implementation of partial spreadability.

3.1. Algebraization of classical probability theory. We review how one arrives at noncommutative notions of probability spaces, random variables and stationary processes when starting from a traditional probabilistic setting.

Let (Ω, Σ, μ) be a standard probability space. Then $\mathcal{L} := L^\infty(\Omega, \Sigma, \mu)$, the Lebesgue space of essentially bounded \mathbb{C} -valued measurable functions, is a commutative von Neumann algebra and, with $f \in \mathcal{L}$, the Lebesgue integral $\text{tr}_\mu(f) := \int_\Omega f d\mu$ is a faithful normal tracial state on \mathcal{L} . In other words, $\text{tr}_\mu(f)$ is the expectation of the essentially bounded \mathbb{C} -valued random variable $f \in \mathcal{L}$ such that $\text{tr}_\mu(f^*f) = \text{tr}_\mu(|f|^2) = 0$ implies $f = 0$ (in the Lebesgue sense). The pair $(\mathcal{L}, \text{tr}_\mu)$ is the standard example for a noncommutative probability space coming from classical probability theory. Conversely, a noncommutative probability space (\mathcal{M}, ψ) with a commutative von Neumann algebra \mathcal{M} can be seen to be isomorphic to this standard example, provided \mathcal{M} has a separable predual.

Now let (Ω_0, Σ_0) be a standard Borel space and consider the (Ω_0, Σ_0) -valued random variable ξ_0 on (Ω, Σ, μ) . Denote by $\mu_0 := \mu \circ \xi_0^{-1}$ the pushforward measure of μ and by $\text{tr}_{\mu_0} := \int_{\Omega_0} \cdot d\mu_0$ the induced (tracial) state on $\mathcal{L}_0 := L^\infty(\Omega_0, \Sigma_0, \mu_0)$. Then $\iota_0(f) := f \circ \xi_0$ defines an injective *-homomorphism from $\mathcal{L}_0 = L^\infty(\Omega_0, \Sigma_0, \mu_0)$ into $\mathcal{L} = L^\infty(\Omega, \Sigma, \mu)$ such that $\text{tr}_\mu \circ \iota_0 = \text{tr}_{\mu_0}$. Altogether we have arrived at the algebraization of the random variable ξ_0 to the noncommutative random variable $\iota_0: (\mathcal{L}_0, \text{tr}_{\mu_0}) \rightarrow (\mathcal{L}, \text{tr}_\mu)$. Note that the noncommutative probability spaces $(\mathcal{L}_0, \text{tr}_{\mu_0})$ and its image under ι_0 in $(\mathcal{L}, \text{tr}_\mu)$ are isomorphic, and hence we frequently identify \mathcal{L}_0 with $L^\infty(\Omega, \Sigma^{\xi_0}, \mu_{\xi_0})$, where $\Sigma^{\xi_0} := \sigma\{\xi_0^{-1}(A) \mid A \in \Sigma_0\}$ and μ_{ξ_0} is the restriction of μ to Σ^{ξ_0} .

Conversely, if $\tilde{\iota}_0: (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ is an injective *-homomorphism, where \mathcal{M} (and thus \mathcal{A}) is a commutative von Neumann algebra with separable predual, then there exist two standard probability spaces (Ω, Σ, μ) and $(\Omega_0, \Sigma_0, \mu_0)$, and an (Ω_0, Σ_0) -valued

random variable ξ_0 on (Ω, Σ, μ) with $\mu \circ \xi_0^{-1} = \mu_0$ such that the noncommutative random variables $\tilde{\iota}_0$ and ι_0 are the same, up to isomorphisms between the involved standard probability spaces.

This algebraization procedure, roughly phrasing, puts the emphasis on the σ -algebra generated by a random variable and less on the random variable itself. Effectively, an (unbounded) random variable ξ_0 may be replaced by an (essentially bounded) random variable $f \circ \xi_0 = \iota_0(f)$, as long as $f \in \mathcal{L}_0$ is chosen such that ξ_0 and $f \circ \xi_0$ generate the same σ -algebra. Of course, this observation for a single random variable $f \circ \xi_0$ extends immediately to multivariate settings, as the family of functions $\{f_i\}_{i \in I} \subset \mathcal{L}_0$ yields the family of (bounded) random variables $\{\iota_0(f_i) = f_i \circ \xi_0\}_{i \in I}$.

We are finally left to address the algebraization procedure for sequences of random variables. Here we will constrain our investigations to sequences given by identically distributed random variables. This simplification improves the transparency of our approach, since it allows us to realize each noncommutative random variable as an injective mapping with the same domain. More explicitly, we will consider a sequence of identically distributed (Ω_0, Σ_0) -valued random variables $\xi \equiv \{\xi_n\}_{n=0}^\infty$ on (Ω, Σ, μ) and denote its canonical local filtration by $\Sigma_\bullet^\xi \equiv \{\Sigma_I^\xi\}_{I \subset \mathbb{N}_0} \subset \Sigma$, where

$$\Sigma_I^\xi := \sigma\{\xi_i^{-1}(A) \mid A \in \Sigma_0, i \in I\}.$$

The σ -algebra $\Sigma_{\mathbb{N}_0}^\xi$ is also denoted simply by Σ^ξ and the measure $\mu|_{\Sigma_{\mathbb{N}_0}^\xi}$ by μ_ξ . Note also that $\Sigma_{[n,n]}^\xi$ is often written as Σ^{ξ_n} or $\sigma\{\xi_n\}$.

We remind that in the setting of the standard probability space (Ω, Σ, μ) , a σ -subalgebra of Σ (generated by random variables) is understood to be completed with respect to sets of μ -measure 0. Since all random variables ξ_n are identically distributed we can identify the pushforward measures $\mu_n = \mu \circ \xi_n^{-1}$ and $\mu_0 = \mu \circ \xi_0^{-1}$. Thus the algebraization procedure yields the sequence of noncommutative random variables

$$\iota \equiv (\iota_n)_{n \geq 0} : (\mathcal{L}_0, \text{tr}_{\mu_0}) \rightarrow (\mathcal{L}, \text{tr}_\mu),$$

where $\iota_n(h) = h \circ \xi_n$. As before, \mathcal{L}_0 can be canonically identified with $L^\infty(\Omega, \Sigma^{\xi_0}, \mu_{\xi_0})$ as a von Neumann subalgebra of \mathcal{L} . Here μ_{ξ_0} denotes the restrictions of μ to Σ^{ξ_0} .

3.2. Partial spreadability and de Finetti theorems in classical probability. Let $\xi \equiv \{\xi_n\}_{n=0}^\infty$ be a sequence of identically distributed (Ω_0, Σ_0) -valued random variables on (Ω, Σ, μ) and denote by $\iota \equiv (\iota_n)_{n \geq 0} : (\mathcal{L}_0, \text{tr}_{\mu_0}) \rightarrow (\mathcal{L}, \text{tr}_\mu)$ the corresponding sequence of noncommutative random variables, obtained from ξ through the algebraization procedure. It is elementary to see that ξ is stationary (in the wide sense) if and only if ι is stationary (in the sense of Definition 2.4.1). Furthermore we may assume $\Sigma = \Sigma_{\mathbb{N}_0}^\xi$ for this stationary sequence. Consequently, there exists a μ -preserving Σ -measurable map η on Ω such that $\xi_n = \xi_0 \circ \eta^n$ for $n \in \mathbb{N}$. This map η lifts to an endomorphism α on the von Neumann algebra \mathcal{L} such that $\alpha(f) = f \circ \eta$ and $\text{tr}_\mu \circ \alpha = \text{tr}_\mu$. Thus the stationary sequence ξ , or its algebraization ι , yields the stationary process $(\mathcal{L}, \text{tr}_\mu, \alpha, \mathcal{L}_0)$ (in the sense of Definition 2.6.1). Conversely, if $(\mathcal{L}, \text{tr}_\mu, \alpha, \mathcal{L}_0)$ is a stationary process with a commutative von Neumann algebra \mathcal{L} acting on a separable Hilbert space, then the

endomorphism α can be implemented by a measure-preserving measurable map η as described before.

This algebraization procedure also applies to distributional invariance principles other than stationarity, like exchangeability or spreadability. Taking into account their characterisations in Proposition 2.4.2, the extended de Finetti theorem can be casted as follows, adapting in wide parts the formulation in Kallenberg's monograph [52].

Theorem 3.2.1. *Suppose $\xi \equiv (\xi_n)_{n \geq 0}$ is a sequence of (Ω_0, Σ_0) -valued random variables on the standard probability space (Ω, Σ, μ) . Then the following are equivalent:*

- (a) ξ is exchangeable;
- (b) ξ is spreadable;
- (b') there exist Σ^ξ -measurable μ_ξ -preserving maps $\{\eta_n\}_{n \geq 0}$ on Ω satisfying the relations $\eta_k \eta_\ell = \eta_{\ell+1} \eta_k$ for $0 \leq k \leq \ell < \infty$ such that $\xi_0 = \xi_0 \circ \eta_k$ for all $k > 0$ (localization) and $\xi_n = \xi_0 \circ \eta_0^n$ for all $n \geq 0$ (stationarity);
- (c) ξ is stationary and conditionally independent.
- (d) ξ is identically distributed and conditionally independent.

Here the conditioning is taken with respect to the tail σ -algebra of the sequence ξ .

The equivalence of conditions (a) and (d) is the content of the traditional de Finetti theorem, and the equivalent characterization by condition (b) is attributed to Ryll-Nardzewski. We refer the reader to [52] for further information of this extended version of the de Finetti theorem and note that, usually, the clearly equivalent condition (c) is omitted in the published literature. The reformulation of spreadability by condition (b') emerges from [53, 31] where it was discovered that spreadability connects to the representation theory of the partial shifts monoid S^+ (see Subsection 2.2), and that this monoid S^+ appears in the inductive limit of the category of semicosimplicial probability spaces.

As spreadability can be implemented through representations of the partial shifts monoid S^+ , it is natural to implement a new distributional invariance principle through representations of the Thompson monoid F^+ , as done in our framework of noncommutative probability. From the view point of classical probability, it is of interest to further investigate the following reformulations of this new distributional invariance principle in noncommutative probability.

Definition 3.2.2. Let $\xi \equiv (\xi_n)_{n \geq 0}$ be a sequence of (Ω_0, Σ_0) -valued random variables on the standard probability space (Ω, Σ, μ) .

- (i) ξ is said to be *partially spreadable* if there exists Σ^ξ -measurable μ_ξ -preserving maps $\{\eta_n\}_{n \geq 0}$ on Ω satisfying the relations $\eta_k \eta_\ell = \eta_{\ell+1} \eta_k$ for $0 \leq k < \ell < \infty$ such that $\xi_0 = \xi_0 \circ \eta_k$ for all $k > 0$ (localization) and $\xi_n = \xi_0 \circ \eta_0^n$ for all $n \geq 0$ (stationarity).
- (ii) The partially spreadable sequence ξ from (i) is said to be *maximal partially spreadable* if $\sigma\{\xi_0\} = \bigcap_{k \geq 1} \mathcal{I}_{\eta_k}$. Here \mathcal{I}_{η_k} denotes the σ -subalgebra of μ -almost η_k -invariant sets in Σ .

More generally, a sequence $\tilde{\xi} \equiv (\tilde{\xi}_n)_{n \geq 0}$ of (Ω_0, Σ_0) -valued random variables on a probability space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ is said to be *(maximal) partially spreadable* if it has the same distribution as a (maximal) partially spreadable sequence ξ .

Thus partial spreadability is implemented by a representation of the Thompson monoid F^+ in the measure-preserving measurable maps on the probability space (Ω, Σ, μ) . It is worthwhile to point out the subtle fact that this representation may not restrict to the probability space $(\Omega, \Sigma^\xi, \mu_\xi)$, where Σ^ξ denotes here the σ -algebra generated by ξ and μ_ξ the restriction of μ to Σ^ξ . This is in contrast to exchangeability, spreadability and stationarity, which are robust under such restrictions to the minimal setting of the underlying probability space.

We provide next the classical reformulation of a local Markov filtration from Definition 2.5.7, as it will be used to generalize the extended de Finetti theorem, Theorem 3.2.1. We stress again that Markovianity of a local filtration, as a property of a (partially ordered) family of σ -subalgebras indexed by ‘time intervals’ $I \subset \mathbb{N}_0$, is a priori not in reference to any stochastic process.

Notation 3.2.3. Suppose $\tilde{\Sigma}$ is a σ -subalgebra of Σ . Then $\mathbb{E}_{\tilde{\Sigma}}$ denotes the μ -preserving (normal) conditional expectation from $L^\infty(\Omega, \Sigma, \mu)$ onto $L^\infty(\Omega, \tilde{\Sigma}, \mu|_{\tilde{\Sigma}})$.

Commonly a filtration of a probability space is understood to be a non-decreasingly (or non-increasingly) totally ordered family of σ -subalgebras of Σ [73]. As we will need to consider families of σ -subalgebras which are indexed by partially ordered sets, we will refer to such families as ‘local filtrations’, a well-established notion in the context of random fields. Recall that $\mathcal{I}(\mathbb{N}_0)$ denotes the set of ‘intervals’ in \mathbb{N}_0 (see Notation 2.5.3).

Definition 3.2.4. A local filtration $\mathcal{F}_\bullet \equiv \{\mathcal{F}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)} \subset \Sigma$ of (Ω, Σ, μ) is said to be *Markovian* (or a *local Markov filtration*) if $\mathcal{F}_{[0, n-1]}$ and $\mathcal{F}_{[n+1, \infty)}$ are conditionally independent over $\mathcal{F}_{[n, n]}$ for all $n \geq 1$. In other words, it holds the Markov property

$$\mathbb{E}_{\mathcal{F}_{[0, n-1]} \vee \mathcal{F}_{[n, n]}} \mathbb{E}_{\mathcal{F}_{[n, n]} \vee \mathcal{F}_{[n+1, \infty)}} = \mathbb{E}_{\mathcal{F}_{[n, n]}} \quad \text{for all } n \geq 1. \quad (\text{M}_c)$$

Here $\mathcal{F}_I \vee \mathcal{F}_J$ denotes the σ -algebra generated by \mathcal{F}_I and \mathcal{F}_J .

An apparently slightly stronger definition of Markovianity, corresponding to saturated Markovianity in Definition 2.5.7, is given by the conditions

$$\mathbb{E}_{\mathcal{F}_{[0, n]}} \mathbb{E}_{\mathcal{F}_{[n, \infty)}} = \mathbb{E}_{\mathcal{F}_{[n, n]}} \quad \text{for all } n \geq 0. \quad (\text{M}'_c)$$

It is easily verified that (M'_c) implies (M_c) , using the isotony properties of local filtrations, see also Lemma 2.5.8. Both Markov conditions (M'_c) and (M_c) are equivalent if the local filtration is *locally minimal*, i.e. it holds $\mathcal{F}_I \vee \mathcal{F}_J = \mathcal{F}_K$ for any $I, J, K \in \mathcal{I}(\mathbb{N}_0)$ with $I \cup J = K$.

In the following we will not distinguish explicitly between the Markov properties (M_c) and (M'_c) , as we will throughout establish the stronger property (M'_c) which is anyway equivalent to (M_c) for locally minimal local filtrations.

Remark 3.2.5. Suppose the local filtration $\mathcal{G}_\bullet \equiv \{\mathcal{G}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)} \subset \Sigma$ is coarser than the local Markov filtration \mathcal{F}_\bullet . Then \mathcal{G}_\bullet is not necessarily Markovian. A sufficient condition so that \mathcal{G}_\bullet inherits Markovianity from \mathcal{F}_\bullet is

$$\mathbb{E}_{\mathcal{G}_{[0, \infty)}} \mathbb{E}_{\mathcal{F}_{[n, n]}} = \mathbb{E}_{\mathcal{G}_{[n, n]}} \quad (n \in \mathbb{N}_0),$$

as it is easily verified:

$$\begin{aligned} \mathbb{E}_{\mathcal{G}_{[0,n]}} \mathbb{E}_{\mathcal{G}_{[n,\infty)}} &= \mathbb{E}_{\mathcal{G}_{[0,n]}} \mathbb{E}_{\mathcal{F}_{[0,n]}} \mathbb{E}_{\mathcal{F}_{[n,\infty)}} \mathbb{E}_{\mathcal{G}_{[n,\infty)}} = \mathbb{E}_{\mathcal{G}_{[0,n]}} \mathbb{E}_{\mathcal{F}_{[n,n]}} \mathbb{E}_{\mathcal{G}_{[n,\infty)}} \\ &= \mathbb{E}_{\mathcal{G}_{[0,n]}} \mathbb{E}_{\mathcal{G}_{[0,\infty)}} \mathbb{E}_{\mathcal{F}_{[n,n]}} \mathbb{E}_{\mathcal{G}_{[n,\infty)}} = \mathbb{E}_{\mathcal{G}_{[0,n]}} \mathbb{E}_{\mathcal{G}_{[n,n]}} \mathbb{E}_{\mathcal{G}_{[n,\infty)}} = \mathbb{E}_{\mathcal{G}_{[n,n]}}. \end{aligned}$$

Definition 3.2.6. Suppose the standard probability space (Ω, Σ, μ) is equipped with a local filtration $\mathcal{F}_\bullet \equiv \{\mathcal{F}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)} \subset \Sigma$. Let $\xi \equiv (\xi_n)_{n \in \mathbb{N}_0}$ be a sequence of (Ω_0, Σ_0) -valued random variables on (Ω, Σ, μ) with canonical local filtration $\Sigma_\bullet^\xi \subset \Sigma$.

- (i) ξ is said to be *adapted to (the local filtration) \mathcal{F}_\bullet* , if $\Sigma_I^\xi \subset \mathcal{F}_I$ for all $I \in \mathcal{I}(\mathbb{N}_0)$.
- (ii) ξ is said to be *\mathcal{F}_\bullet -Markovian* (or an *\mathcal{F}_\bullet -Markov sequence*), if ξ is adapted to \mathcal{F}_\bullet and

$$\mathbb{E}_{\mathcal{F}_{[0,n]}} \mathbb{E}_{\sigma\{\xi_{n+1}\}} = \mathbb{E}_{\sigma\{\xi_n\}} \mathbb{E}_{\sigma\{\xi_{n+1}\}} \quad (3.2.1)$$

A Σ_\bullet^ξ -Markovian sequence ξ is just said to be *Markovian* (or a *Markov sequence*).

\mathcal{F}_\bullet -Markovianity of a sequence makes use only of the totally ordered family $\{\mathcal{F}_{[0,n]}\}_{n \geq 0}$ in \mathcal{F}_\bullet which is a filtration in the common sense. This gives the usual notion of an $\{\mathcal{F}_{[0,n]}\}_{n \geq 0}$ -Markovian sequence, see for example [51].

Lemma 3.2.7. Let $\xi \equiv (\xi_n)_{n \in \mathbb{N}_0}$ be a sequence of (Ω_0, Σ_0) -valued random variables on (Ω, Σ, μ) with canonical local filtration $\Sigma_\bullet^\xi \subset \Sigma$. Then the following are equivalent:

- (a) ξ is a Markov sequence;
- (b) Σ_\bullet^ξ is a local Markov filtration.

Proof. The implication ‘(b) \Rightarrow (a)’ is clear by taking $\mathcal{F}_\bullet \equiv \Sigma_\bullet^\xi$ in (3.2.1). The converse implication ‘(a) \Rightarrow (b)’ follows immediately from Proposition 2.6.13 and Remark 2.6.14, as it formulates the same in the language of noncommutative probability. \square

Suppose now that the standard probability space (Ω, Σ, μ) is equipped with a representation ρ_* of the Thompson monoid $F^+ = \langle g_0, g_1, g_2, \dots \mid g_k g_\ell = g_{\ell+1} g_k \text{ for } 0 \leq k < \ell < \infty \rangle^+$ in the μ -preserving Σ -measurable maps on Ω . Let us put $\eta_k := \rho_*(g_k)$ for brevity and recall that \mathcal{I}_{η_k} denotes the σ -subalgebra of μ -almost η_k -invariant sets in Σ . Putting $\mathcal{J}_n := \bigcap_{k \geq n+1} \mathcal{I}_{\eta_k}$ we obtain the tower of σ -subalgebras

$$\mathcal{J}_{-1} \subset \mathcal{J}_0 \subset \mathcal{J}_1 \subset \dots \subset \mathcal{J}_\infty := \sigma\{\mathcal{J}_n \mid n \geq 0\} \subset \Sigma.$$

Note that \mathcal{J}_{-1} is the σ -subalgebra of μ -almost $\rho_*(F^+)$ -invariant sets in Σ . This tower is a filtration which can be further upgraded to the local filtration $\mathcal{F}_\bullet^+ \equiv \{\mathcal{F}_I^+\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ by putting

$$\mathcal{F}_{[0,n]}^+ := \mathcal{J}_n, \quad \mathcal{F}_{[m,m+n]}^+ := \eta_0^m(\mathcal{J}_n), \quad \mathcal{F}_{[m,\infty)}^+ := \eta_0^m(\mathcal{J}_\infty),$$

where $\eta_0(\mathcal{J}) := \sigma\{\eta_0(J) \mid J \in \mathcal{J}\}$ for a σ -subalgebra $\mathcal{J} \subset \Sigma$.

We will assume for the remainder of this subsection that the representation ρ_* of the Thompson monoid F^+ , and thus the local filtration \mathcal{F}_\bullet^+ , has the so-called generating property $\mathcal{J}_\infty = \mathcal{F}_{\mathbb{N}_0}^+ = \Sigma$. This can always be achieved by restriction of σ -algebra and measure of the standard probability space (Ω, Σ, μ) when necessary, as shown in Subsection 4.1 in the general noncommutative setting. This ensures, by an application of the mean ergodic theorem, that $\mathcal{J}_n = \mathcal{I}_{\eta_{n+1}}$ for every $n \geq -1$. We obtain as a main

result that the fixed point σ -algebras $\{\mathcal{J}_n\}_{n \geq 0}$ of the represented Thompson monoid F^+ encode Markovianity.

Theorem 3.2.8. *The local filtration \mathcal{F}_\bullet^+ is Markovian (in the sense of Definition 3.2.4).*

Actually this result is the classical reformulation of Corollary 4.2.4 which emerges from applications of the von Neumann mean ergodic theorem in Theorem 4.2.2. The former establishes, here reformulated in the language of probability theory, that the two σ -subalgebras \mathcal{J}_{m+k} and $\eta_0^k(\mathcal{J}_n)$ are conditionally independent over $\eta_0^k(\mathcal{J}_m)$ for $0 \leq m \leq n < \infty$ and $k \geq 0$. Already some instances of these conditional independences between shifted fixed point σ -algebras suffice to encode the Markovianity of the local filtration \mathcal{F}_\bullet^+ . We refer the reader to Subsection 4.2 for a more detailed investigation of these conditional independences in the general setting of noncommutative probability.

Partial spreadability as a new distributional invariance principle of a sequence ξ (as casted in Definition 3.2.2) provides a sufficient condition to ensure that the canonical local filtration of ξ is adapted to the local filtration of fixed point σ -algebras \mathcal{F}_\bullet^+ . Altogether we arrive at the following result which takes into account the stationarity of a partially spreadable sequence, as evident from Definition 3.2.2 of this distributional invariance principle. For clarity of its formulation, the sequence ξ and the representation ρ_* of the Thompson monoid F^+ are assumed to be realized on the same probability space.

Theorem 3.2.9. *A partially spreadable sequence ξ is stationary and adapted to the local Markov filtration \mathcal{F}_\bullet^+ .*

For the convenience of the reader from classical probability, let us we sketch the proof which essentially follows the arguments from Remark 4.3.2 in the general setting of noncommutative probability.

Proof. Since the sequence $\xi \equiv (\xi_n)_{n \geq 0}$ and the representation ρ_* of the Thompson monoid F^+ are assumed to be realized on the the same standard probability space (Ω, Σ, μ) , partial spreadability ensures $\xi_0 \circ \eta_n = \xi_0$ and $\xi_n = \xi_0 \circ \rho_*(g_0^n) = \xi_0 \circ \eta_0^n$ for all $n \geq 1$. The latter condition implies the stationarity of the sequence. As the local filtration \mathcal{F}_\bullet^+ is Markovian by Theorem 3.2.8, we are left to verify the adaptedness of the sequence ξ to \mathcal{F}_\bullet^+ . Indeed, it follows from the localization properties $\xi_0 \circ \eta_n = \xi_0$ (for $n \geq 1$) that $\sigma\{\xi_0\} \subset \bigcap_{k \geq 1} \mathcal{I}_{\eta_k} = \mathcal{J}_0$. Using the relations of the generators of F^+ , one concludes further that $\Sigma_{[0,n]}^\xi \subset \mathcal{J}_n$ for every $n \geq 0$. This in turn gives the inclusions $\Sigma_{[m,n]}^\xi \subset \eta_0^m(\mathcal{J}_{n-m}) = \mathcal{F}_{[m,n]}^+$ for all $0 \leq m \leq n$ and, by an approximation argument, $\Sigma_{[m,\infty]}^\xi \subset \mathcal{F}_{[m,\infty]}^+$ for all $m \geq 0$. Overall, we arrive at $\Sigma_I^\xi \subset \mathcal{F}_I^+$ for all $I \in \mathcal{I}(\mathbb{N}_0)$, that is, the adaptedness of the sequence ξ to the local filtration \mathcal{F}_\bullet^+ . \square

We stress that a partially spreadable sequence ξ may not be Markovian. The inclusions of σ -subalgebras

$$\begin{aligned} \mathcal{F}_{[n,n]}^+ &\subset \mathcal{F}_{[0,\infty]}^+ \\ \cup &\cup \\ \sigma\{\xi_n\} &\subset \Sigma^\xi \end{aligned} \tag{3.2.2}$$

are always valid for all $n \geq 0$, but may not ensure that the sequence ξ is Markovian. A sufficient condition for the latter is that Σ^ξ and $\mathcal{F}_{[n,n]}^+$ are conditionally independent over $\sigma\{\xi_n\}$ for all $n \geq 0$, for example. Such conditions, ensuring Markovianity of a sequence, are studied in greater detail in Subsection 4.3 in the operator algebraic framework of commuting squares.

All inclusions (3.2.2) are valid of course in the particular situation where the canonical local filtration Σ_\bullet^ξ of the sequence and the local Markov filtration \mathcal{F}_\bullet^+ coincide, i.e. $\Sigma_I^\xi = \mathcal{F}_I^+$ for all ‘intervals’ $I \in \mathcal{I}(\mathbb{N}_0)$. Combining Lemma 3.2.7 and Theorem 3.2.9 one arrives immediately at the following result.

Corollary 3.2.10. *A partially spreadable sequence ξ is Markovian if $\Sigma_\bullet^\xi \equiv \mathcal{F}_\bullet^+$.*

Actually Theorem 3.2.9 can be strengthened by considering the local filtration $\mathcal{G}_\bullet^+ \equiv \{\mathcal{G}_I^+ := \sigma\{\eta_0^i(\mathcal{J}_0) \mid i \in I\}\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ which is coarser than \mathcal{F}_\bullet^+ .

Theorem 3.2.11. *A partially spreadable sequence ξ is stationary and adapted to the local Markov filtration \mathcal{G}_\bullet^+ .*

The proof of this strengthened result is given in Subsection 4.3 in the noncommutative framework. As before, one needs to stipulate additional conditions to ensure that ξ is Markovian. Sufficient conditions are now obtained by replacing \mathcal{F}_\bullet^+ by \mathcal{G}_\bullet^+ in (3.2.2) and stipulating conditional independence over $\sigma\{\xi_n\}$ of Σ^ξ and $\mathcal{G}_{[n,n]}^+$ for all $n \geq 0$. Actually these conditions can be further strengthened so that one arrives at the following result which essentially features Theorem 4.3.4 in the traditional language of probability theory.

Theorem 3.2.12. *Suppose the partially spreadable sequence ξ is such that the σ -algebras Σ^ξ and \mathcal{J}_0 are conditionally independent over $\sigma\{\xi_0\}$. Then ξ is a stationary Markov sequence.*

Clearly this conditional independence condition in above theorem is satisfied whenever $\sigma(\xi_0) = \mathcal{J}_0$. This is precisely the additional condition which upgrades ‘partial spreadability’ to ‘maximal partial spreadability’, as an inspection of Definition 3.2.2 easily reveals.

The above results have not so far addressed whether a stationary Markov sequence can be seen to be (maximal) partially spreadable. An affirmative answer to this question in classical probability requires the construction of a representation of the Thompson monoid F^+ from a stationary Markov sequence. An intermediate step towards the construction of such a representation is provided by the construction of a certain Markov dilation, as it is discussed in Subsection 3.4 below. Altogether this allows us to establish a de Finetti theorem for stationary Markov sequences in Theorem 5.2.5 which was stated in its classical reformulation as Theorem 1.0.1. We state it here again for the convenience of the reader.

Theorem. *Let $\xi \equiv (\xi_n)_{n \in \mathbb{N}_0}$ be a sequence of (Ω_0, Σ_0) -valued random variables on the standard probability space (Ω, Σ, μ) . Then the following are equivalent:*

- (a) ξ is maximal partially spreadable;
- (b) ξ is a stationary Markov sequence.

A proof of this theorem is obtained from transferring the proof of Theorem 5.2.5 to the traditional setting in terms of classical random variables.

Remark 3.2.13. The extended de Finetti theorem characterizes spreadable sequences of random variables as mixtures of sequences of conditionally independent, identically distributed random variables. A similar complete characterization of partially spreadable sequences of random variables is presently an open problem. It is tempting to speculate that any stationary sequence ξ is partially spreadable. Such a conjecture is supported by the fact that a function of a stationary Markov sequence may fail to be Markovian. On the other hand, a function of a partially spreadable sequence yields again a partially spreadable sequence. We will further discuss this phenomenon in Subsection 3.3. Finally, let us briefly remark that mixtures of stationary Markov sequences may also be considered to emerge from certain functions of a Markov sequence.

3.3. Functions of classical stationary Markov sequences and their algebraization. We motivate why partial spreadability characterizes a much larger class of stationary stochastic processes than that of stationary Markov sequences. For this purpose, consider the sequence $\zeta \equiv \{\zeta_n := f \circ \xi_n\}_{n=0}^\infty$ for some (measurable) function $f : (\Omega_0, \Sigma_0) \rightarrow (\Omega_0, \Sigma_0)$ of the stationary sequence ξ (as above), and denote by $\Sigma_\bullet^\zeta \equiv \{\Sigma_I^\zeta\}_{I \subset \mathbb{N}_0}$ the canonical local filtration of ζ . Then it is well-known that stationarity and stochastic independence of ξ transfer to the sequence ζ , but Markovianity may not (for example see [13] and [65, page 59]). If the function f is injective then the Markovianity of Σ_\bullet^ξ implies the Markovianity of Σ_\bullet^ζ . However, in general we only get the inclusions $\Sigma_\bullet^\zeta \subset \Sigma_\bullet^\xi \subset \Sigma$. Thus the local filtration Σ_\bullet^ζ may be coarser than the local Markov filtration Σ_\bullet^ξ and, in general, one can not expect the sequence ζ to be Markovian, that is, the canonical local filtration Σ_\bullet^ζ may not be Markovian. But partial spreadability is robust when passing from ξ to ζ , in contrast to Markovianity.

Proposition 3.3.1. *Suppose ξ is partially spreadable. Then ζ is partially spreadable.*

Proof. Since $\zeta_n = f \circ \xi_n$, it follows from the partial spreadability of ξ that $f \circ \xi_0 = f \circ \xi_0 \circ \eta_k$ and $f \circ \xi_n = f \circ \xi_0 \circ \eta_0^n$ for all $k, n > 0$. \square

Consequently, the class of partially spreadable sequences of random variables is much larger than the class of stationary Markov sequences. We recall that (Ω_0, Σ_0) denotes a standard Borel space and (Ω, Σ, μ) a standard probability space.

Theorem 3.3.2. *Let $\xi \equiv (\xi_n)_{n \in \mathbb{N}_0}$ be a sequence of (Ω_0, Σ_0) -valued random variables on (Ω, Σ, μ) . If ξ is a stationary Markov sequence then its function sequence $\zeta \equiv (\zeta_n := f \circ \xi_n)_{n=0}^\infty$ is partially spreadable for any (measurable) function $f : (\Omega_0, \Sigma_0) \rightarrow (\Omega_0, \Sigma_0)$.*

Proof. A stationary Markov sequence is (maximal) partially spreadable by Theorem 1.0.1. Thus Proposition 3.3.1 applies. \square

Remark 3.3.3. Maximal partial spreadability of the sequence ξ may not be inherited by ζ , for the same reason that a function of a Markov sequence may be non-Markovian.

We close this subsection by addressing how passing from the sequence ξ to its function sequence ζ reappears as restricting the corresponding noncommutative random

variables. Continuing our discussion from above, upon the algebraization of classic random variables, the inclusions of local filtrations

$$\Sigma_{\bullet}^{\zeta} \subset \Sigma_{\bullet}^{\xi} \subset \Sigma$$

become inclusions of Lebesgue spaces of essentially bounded functions:

$$L^{\infty}(\Omega, \Sigma^{\zeta}, \mu_{\zeta}) \subset L^{\infty}(\Omega, \Sigma^{\xi}, \mu_{\xi}) \subset L^{\infty}(\Omega, \Sigma, \mu),$$

where μ_{ζ} denotes the restrictions of μ to Σ^{ζ} . Thus taking a function of a sequence of random variables becomes restriction of a sequence of noncommutative random variables. Altogether this entails that a Markovian sequence of noncommutative random variables

$$\iota: (\mathcal{L}_0, \text{tr}_{\mu_0}) \rightarrow (\mathcal{L}, \text{tr}_{\mu})$$

restricts to a sequence

$$\iota^{\zeta}: (\tilde{\mathcal{L}}_0, \text{tr}_{\tilde{\mu}_0}) \rightarrow (\mathcal{L}, \text{tr}_{\mu}),$$

which may fail to be Markovian with respect to its canonical local filtration. Here we have put $\tilde{\mathcal{L}}_0 := L^{\infty}(\Omega, \Sigma^{\zeta_0}, \tilde{\mu}_0) \subset \mathcal{L}_0 = L^{\infty}(\Omega, \Sigma^{\xi_0}, \mu_0)$, where $\tilde{\mu}_0$ denotes the restriction of $\mu_0 := \mu_{\xi_0}$ from Σ^{ξ_0} to Σ^{ζ_0} . Consequently, in reference to Definition 3.2.4, the Markovianity of the local filtration Σ_{\bullet}^{ξ} may not pass to the local filtration Σ_{\bullet}^{ζ} . Of course, this may be reformulated in terms of commuting squares of commutative von Neumann algebras, see Remark 2.5.9.

3.4. A result on Markov dilations in classical probability theory. Presently it is an open problem if a representation of the Thompson monoid F^+ can be constructed directly based on the usual Daniell-Kolmogorov construction for stationary Markov processes. This issue is bypassed by rewriting a Markov process as an open dynamical system, a construction which is well-known within quantum probability [1]. Here we review this construction in the framework of classical probability along the presentation given in the habilitation thesis of Kümmerer [57]. This construction is underlying the construction of a representation of the Thompson monoid F^+ in Subsection 5.2.

Notation 3.4.1. For the remainder of this subsection the noncommutative probability space $(\mathcal{L}, \text{tr}_{\lambda})$ is given by the Lebesgue space of essentially bounded functions $\mathcal{L} := L^{\infty}([0, 1], \lambda)$ and $\text{tr}_{\lambda} := \int_{[0,1]} \cdot d\lambda$ as the faithful normal state on \mathcal{L} . Here λ denotes the Lebesgue measure on the unit interval $[0, 1] \subset \mathbb{R}$.

Theorem 3.4.2 ([57, 4.4.2]). *Let T be a φ -Markov map on \mathcal{A} , where \mathcal{A} is a commutative von Neumann algebra with separable predual. Then there exists $\alpha \in \text{End}(\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_{\lambda})$ such that $(\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_{\lambda}, \alpha, \iota_0)$ is a Markov dilation of T . That is, $(\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_{\lambda}, \alpha, \mathcal{A} \otimes \mathbb{1}_{\mathcal{L}})$ is a stationary Markov process, and for all $n \in \mathbb{N}_0$,*

$$T^n = \iota_0^* \alpha^n \iota_0,$$

where $\iota_0: (\mathcal{A}, \varphi) \rightarrow (\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_{\lambda})$ denotes the canonical embedding $\iota_0(a) = a \otimes \mathbb{1}_{\mathcal{L}}$ such that $E_0 := \iota_0 \circ \iota_0^*$ is the $\varphi \otimes \text{tr}_{\lambda}$ -preserving normal conditional expectation from $\mathcal{A} \otimes \mathcal{L}$ onto $\mathcal{A} \otimes \mathbb{1}_{\mathcal{L}}$.

For the convenience of the reader we outline next a proof of this folklore result in noncommutative probability. Our arguments follow almost verbatim those given by Kümmerer in his habilitation thesis [57], adapting therein arguments from the setting of bilateral to that of unilateral noncommutative stationary processes.

Proof. The Daniell-Kolmogorov construction gives a dilation $(\mathcal{M}, \psi, \beta, \kappa)$ of the φ -Markov operator T on \mathcal{A} in the following way. A state ψ_{alg} on the infinite algebraic tensor product $\mathcal{M}_{\text{alg}} := \odot_{\mathbb{N}_0} \mathcal{A}$ is defined through

$$\psi_{\text{alg}}(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes \mathbb{1}_{\mathcal{A}} \otimes \cdots) = \varphi(a_0 T(a_1 T(a_2 \cdots T(a_{n-1} T(a_n)) \cdots))).$$

Next one applies the GNS construction to the pair $(\mathcal{M}_{\text{alg}}, \psi_{\text{alg}})$, a so-called $*$ -algebraic probability space, to obtain the noncommutative probability space (\mathcal{M}, ψ) (as introduced in Subsection 2.3) such that $\mathcal{M} = \Pi_{\psi}(\mathcal{M}_{\text{alg}})''$ and $\psi(\Pi_{\psi}(x)) = \langle 1_{\psi}, \Pi_{\psi}(x) 1_{\psi} \rangle = \psi_{\text{alg}}(x)$. Here $(\mathcal{H}_{\psi}, \Pi_{\psi}, 1_{\psi})$ denotes the GNS triple of $(\mathcal{M}_{\text{alg}}, \psi_{\text{alg}})$ and $\Pi_{\psi}(\mathcal{M}_{\text{alg}})''$ the double commutant of $\Pi_{\psi}(\mathcal{M}_{\text{alg}})$ in $\mathcal{B}(\mathcal{H}_{\psi})$. We remark that the normal state ψ on \mathcal{M} is automatically faithful, as ψ is tracial. The right shift defined on $\mathcal{M}_{\text{alg}} = \odot_{\mathbb{N}_0} \mathcal{A}$ by $a_0 \otimes a_1 \otimes \cdots \mapsto \mathbb{1}_{\mathcal{A}} \otimes a_0 \otimes a_1 \otimes \cdots$ extends in the GNS representation to an endomorphism β of (\mathcal{M}, ψ) such that $\beta(\Pi_{\psi}(a_0 \otimes a_1 \otimes \cdots)) = \Pi_{\psi}(\mathbb{1}_{\mathcal{A}} \otimes a_0 \otimes a_1 \otimes \cdots)$. Moreover $\kappa(a) := \Pi_{\psi}(a \otimes \mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \otimes \cdots)$ defines a noncommutative random variable κ from (\mathcal{A}, φ) into (\mathcal{M}, ψ) such that $(\mathcal{M}, \psi, \beta, \kappa)$ is a minimal Markov dilation of the φ -Markov map T on \mathcal{A} . We stress here that the endomorphism β is simply the right shift on tensor products, and all information about the Markov map T is encoded into the state ψ .

As $(\mathcal{M}, \psi, \beta, \kappa)$ is a dilation of the φ -Markov map T on \mathcal{A} , so is its amplification to $(\mathcal{M} \otimes \mathcal{L}, \psi \otimes \text{tr}_{\lambda}, \beta \otimes \text{Id}_{\mathcal{L}}, \kappa_{\mathcal{M}} \circ \kappa)$, where $\kappa_{\mathcal{M}}$ is the canonical embedding of \mathcal{M} into $\mathcal{M} \otimes \mathcal{L}$. We show next that the two noncommutative probability spaces $(\mathcal{M} \otimes \mathcal{L}, \psi \otimes \text{tr}_{\lambda})$ and $(\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_{\lambda})$ are isomorphic. More explicitly, we will show the existence of an isomorphism $\pi: \mathcal{M} \otimes \mathcal{L} \rightarrow \mathcal{A} \otimes \mathcal{L}$ such that $\pi(\kappa(a) \otimes \mathbb{1}_{\mathcal{L}}) = a \otimes \mathbb{1}_{\mathcal{L}} = \iota_0(a)$ and $\psi \otimes \text{tr}_{\lambda} = (\varphi \otimes \text{tr}_{\lambda}) \circ \pi$. Consequently $\alpha := \pi \circ \beta \circ \pi^{-1}$ will define an endomorphism of $(\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_{\lambda})$ such that $(\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_{\lambda}, \alpha, \iota_0)$ is a Markov dilation of the φ -Markov map T on \mathcal{A} .

As \mathcal{A} is a commutative von Neumann algebra with separable predual, it can be identified with $L^{\infty}(\Gamma, \mu)$ where Γ is a standard Borel space and μ is a probability measure on Γ which induces φ (that is, $\varphi = \int_{\Gamma} \cdot d\mu$). Moreover \mathcal{M} also has separable predual, since $\mathcal{M} = \bigvee \{\beta^n \kappa(\mathcal{A}) \mid n \in \mathbb{N}_0\}$ by the minimality of the dilation. Consequently we can assume that \mathcal{M} is faithfully represented on a separable Hilbert space \mathcal{H} with cyclic separating vector ξ inducing the state ψ .

Applying the theory of disintegration ([70, Chapter IV]), we obtain a measurable field of commutative von Neumann algebras $\{\{\mathcal{M}(\gamma), \mathcal{H}(\gamma)\} : \gamma \in \Gamma\}$ such that

$$\{\mathcal{M}, \mathcal{H}\} = \int_{\Gamma}^{\oplus} \{\mathcal{M}(\gamma), \mathcal{H}(\gamma)\} d\mu(\gamma),$$

with $\kappa(\mathcal{A}) \cong L^{\infty}(\Gamma, \mu)$ as the diagonal subalgebra. Let

$$\xi = \int_{\Gamma}^{\oplus} \xi(\gamma) d\mu(\gamma), \quad \psi = \int_{\Gamma}^{\oplus} \psi(\gamma) d\mu(\gamma)$$

be the corresponding desintegration of the vector ξ and the state ψ (see [70, Chapter IV, Definitions 8.14, 8.15, 8.33]). It follows from the uniqueness of desintegration ([70, Chapter IV, Theorem 8.34] that, for $x \in \mathcal{A}(\gamma)$,

$$\varphi(\gamma)(x) = \langle \xi(\gamma), x\xi(\gamma) \rangle$$

for μ -almost all $\gamma \in \Gamma$. In particular, $\xi(\gamma)$ is cyclic and separating for μ -almost all $\gamma \in \Gamma$. Similarly, with $\mathcal{L} = L^\infty([0, 1], \lambda)$ and $\mathcal{K} = L^2([0, 1], \lambda)$, one has the unique desintegration

$$\{\mathcal{M} \otimes \mathcal{L}, \mathcal{H} \otimes \mathcal{K}\} = \int_{\Gamma}^{\oplus} \{\mathcal{M}(\gamma) \otimes \mathcal{L}, \mathcal{H}(\gamma) \otimes \mathcal{K}\} d\mu(\gamma),$$

and

$$\psi \otimes \text{tr}_{\lambda} = \int_{\Gamma}^{\oplus} \psi(\gamma) \otimes \text{tr}_{\lambda} d\mu(\gamma).$$

For almost all $\gamma \in \Gamma$, the noncommutative probability space $(\mathcal{M}(\gamma) \otimes \mathcal{L}, \varphi(\gamma) \otimes \text{tr}_{\lambda})$ has a von Neumann algebra without atoms acting on a separable Hilbert space. Hence, $\mathcal{M}(\gamma) \otimes \mathcal{L}$ is isomorphic to \mathcal{L} for almost all $\gamma \in \Gamma$ ([70, Chapter III, Theorem 1.22]). Under this isomorphism, the state $\psi(\gamma) \otimes \text{tr}_{\lambda}$ is mapped into some faithful normal state on \mathcal{L} and we can even find an isomorphism mapping $\psi(\gamma) \otimes \text{tr}_{\lambda}$ into tr_{λ} (see [66, Theorem 15.3.9]). Altogether we get an isomorphism $\pi(\gamma)$ between $(\mathcal{M}(\gamma) \otimes \mathcal{L}, \psi(\gamma) \otimes \text{tr}_{\lambda})$ and $(\mathcal{L}, \text{tr}_{\lambda})$ for almost all $\gamma \in \Gamma$. In particular, $\pi(\gamma)$ is even a spatial isomorphism between the pairs $\{\mathcal{M}(\gamma) \otimes \mathcal{L}, \mathcal{H}(\gamma) \otimes \mathcal{K}\}$ and $\{\mathcal{L}, \mathcal{K}\}$. Therefore, applying [70, Section IV, Theorem 8.28], we obtain an isomorphism between

$$\left(\int_{\Gamma}^{\oplus} \{\mathcal{M}(\gamma) \otimes \mathcal{L}, \mathcal{H}(\gamma) \otimes \mathcal{K}\} d\mu(\gamma), \int_{\Gamma}^{\oplus} \psi(\gamma) \otimes \text{tr}_{\lambda} d\mu(\gamma) \right)$$

and

$$\left(\int_{\Gamma}^{\oplus} \{\mathcal{L}, \mathcal{K}\} d\mu(\gamma), \int_{\Gamma}^{\oplus} \text{tr}_{\lambda} d\mu(\gamma) \right)$$

which, by [70, Chapter IV, Theorem 8.30], in turn is isomorphic to

$$(L^\infty(\Gamma, \mu) \otimes \mathcal{L}, \lambda), \varphi \otimes \text{tr}_{\lambda}) = (\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_{\lambda}).$$

This establishes the existence of an isomorphism π between the noncommutative probability spaces $(\mathcal{M} \otimes \mathcal{L}, \psi \otimes \text{tr}_{\lambda})$ and $(\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_{\lambda})$. \square

As pointed out by Kümmerer in [57], Theorem 3.4.2 ensures the existence of a Markov dilation, but does not provide an explicit construction. Also, its proof rests on the traditional Daniell-Kolmogorov construction for a stationary process. But a concrete construction is available for stationary Markov chains with values in a finite set which, moreover, is independent of the Daniell-Kolmogorov construction. For the convenience of the reader, we demonstrate this for the algebraic reformulation of a stationary $\{0, 1\}$ -valued Markov chain, reproducing the arguments of Kümmerer in [57, 4.4.3].

Example 3.4.3. An algebraic model for a coin toss is given by the noncommutative probability space $(\mathbb{C}^2, \varphi_q)$, where \mathbb{C}^2 represents the two-dimensional unital $*$ -algebra of all \mathbb{C} -valued functions on the set $\{0, 1\}$, and for some fixed $q \in (0, 1)$,

$$\varphi_q(\mathbf{a}) = qa_1 + (1 - q)a_2, \quad \text{with } \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{C}^2,$$

defines a faithful (tracial) state φ_q on \mathbb{C}^2 . Now consider the transition matrix $T: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by

$$T = \begin{bmatrix} 1 - p_1 & p_1 \\ p_2 & 1 - p_2 \end{bmatrix}$$

for some fixed $p_1, p_2 \in [0, 1]$. For some fixed $0 < q < 1$, this map T is a φ_q -Markov map (in the sense of Definition 2.3.2) if and only if the pair (p_1, p_2) satisfies the following two conditions:

- (i) $p_2(1 - q) = p_1q$ (stationarity of the map);
- (ii) either $p_1 = p_2 = 0$ or $p_1, p_2 > 0$ (faithfulness of the state).

Note that $p_1 = p_2 = 0$ corresponds to T being the identity matrix, and $p_1 = p_2 = 1$ to T being the flip, both of which are automorphisms on \mathbb{C}^2 . Thus these two cases yield trivial stationary Markov dilations. For notational convenience we will exclude these two cases in the following discussion.

To start the construction of a stationary Markov dilation (in the sense of Kümmerer), let the noncommutative probability space $(\mathcal{L}, \text{tr}_\lambda)$ be given by

$$\mathcal{L} = L^\infty([0, 1], \lambda) \quad \text{and} \quad \text{tr}_\lambda = \int_{[0, 1]} \cdot d\lambda,$$

where λ denotes the Lebesgue measure on $[0, 1]$. The goal is to construct an automorphism $C \in \text{Aut}(\mathbb{C}^2 \otimes \mathcal{L}, \varphi_q \otimes \text{tr}_\lambda)$ such that $\iota^* \circ C \circ \iota(\mathbf{a}) = T(\mathbf{a})$, where the embedding $\iota: \mathbb{C}^2 \rightarrow \mathbb{C}^2 \otimes \mathcal{L}$ is given by $\iota(\mathbf{a}) = \mathbf{a} \otimes \mathbb{1}_{\mathcal{L}}$. For this purpose, let

$$\begin{aligned} (\Omega_1, \lambda_1) &:= ([0, 1], q\lambda) \\ (\Omega_2, \lambda_2) &:= ([0, 1], (1 - q)\lambda) \end{aligned}$$

and define on the disjoint union $\Omega_1 \sqcup \Omega_2$ the probability measure μ such that

$$\mu(A) = \lambda_1(A \cap \Omega_1) + \lambda_2(A \cap \Omega_2).$$

In the following we canonically identify the two (isomorphic) noncommutative probability spaces $(\mathbb{C}^2 \otimes \mathcal{L}, \varphi_q \otimes \text{tr}_\lambda)$ and $(L^\infty(\Omega_1 \sqcup \Omega_2, \mu), \int_{\Omega_1 \sqcup \Omega_2} \cdot d\mu)$ such that

$$\varphi_q \otimes \text{tr}_\lambda(\mathbf{a} \otimes f) = a_1 \int_{\Omega_1} f d\lambda_1 + a_2 \int_{\Omega_2} f d\lambda_2.$$

Thus, under this identification, the embedding ι reads as

$$\iota(\mathbf{a}) = a_1 \chi_{\Omega_1} + a_2 \chi_{\Omega_2},$$

where χ_A denotes the characteristic function of the measurable set $A \in \Omega_1 \sqcup \Omega_2$, and its adjoint ι^* becomes

$$\iota^*(f) = \begin{bmatrix} \int_{\Omega_1} f d\lambda_1 \\ \int_{\Omega_2} f d\lambda_2 \end{bmatrix}.$$

For the construction of the automorphism C , we further partition

$$\begin{aligned}\Omega_1 &= \Omega_{1,1} \sqcup \Omega_{1,2} := [0, 1 - p_1] \sqcup (1 - p_1, 1], \\ \Omega_2 &= \Omega_{2,1} \sqcup \Omega_{2,2} := [0, p_2] \sqcup [p_2, 1],\end{aligned}$$

to obtain

$$\begin{aligned}\mu(\Omega_{1,1}) &= (1 - p_1)q, & \mu(\Omega_{1,2}) &= p_1q, \\ \mu(\Omega_{2,1}) &= p_2(1 - q), & \mu(\Omega_{2,2}) &= (1 - p_2)(1 - q).\end{aligned}$$

This way of partitioning is motivated by

$$\mu(\Omega_{1,2}) = \mu(\Omega_{2,1}) \iff p_1q = p_2(1 - q),$$

where the righthand side of this equivalence is the concrete form of the stationarity condition $\varphi_q \circ T = \varphi_q$. Thus there exists a bijective μ -preserving measurable map τ on $\Omega_1 \sqcup \Omega_2$ such that

$$\tau(\Omega_{1,1}) = \Omega_{1,1}, \quad \tau(\Omega_{2,2}) = \Omega_{2,2}, \quad \tau(\Omega_{1,2}) = \Omega_{2,1}.$$

We choose τ to be the identity map when restricted to $\Omega_{1,1} \cup \Omega_{2,2}$. Let C denote the automorphism on $L^\infty(\Omega_1 \sqcup \Omega_2, \mu)$ induced by τ , such that $C(\chi_A) = \chi_{\tau(A)}$. We claim that $\iota^* \circ C \circ \iota(\mathbf{a}) = T(\mathbf{a})$. Indeed,

$$\begin{aligned}\iota^* C \iota(\mathbf{a}) &= \iota^* C(a_1 \chi_{\Omega_1} + a_2 \chi_{\Omega_2}) \\ &= \iota^* C(a_1 \chi_{\Omega_{1,1}} + a_1 \chi_{\Omega_{1,2}} + a_2 \chi_{\Omega_{2,1}} + a_2 \chi_{\Omega_{2,2}}) \\ &= \iota^* (a_1 \chi_{\Omega_{1,1}} + a_1 \chi_{\Omega_{2,1}} + a_2 \chi_{\Omega_{1,2}} + a_2 \chi_{\Omega_{2,2}}) \\ &= \begin{bmatrix} a_1(1 - p_1) + a_2 p_1 \\ a_1 p_2 + a_2(1 - p_2) \end{bmatrix} = T(\mathbf{a}).\end{aligned}$$

As the invariance property $\int_{\Omega_1 \sqcup \Omega_2} C(f) d\mu = \int_{\Omega_1 \sqcup \Omega_2} f d\mu$ is evident from the definition of C , we have verified that $C \in \text{Aut}(\mathbb{C}^2 \otimes \mathcal{L}, \varphi_q \otimes \text{tr}_\lambda)$ (after the canonical identification made above).

We are left to provide a probability space (\mathcal{M}, ψ) and an endomorphism $\alpha \in \text{End}(\mathcal{M}, \psi)$ such that $(\mathcal{M}, \psi, \alpha, \iota)$ is a Markov dilation of T . Let $\mathcal{M} := \mathbb{C}^2 \otimes \mathcal{L}^{\otimes \mathbb{N}_0}$ and $\psi := \varphi_q \otimes \text{tr}_\lambda^{\otimes \mathbb{N}_0}$. Then the endomorphism α on $\mathbb{C}^2 \otimes \mathcal{L}^{\otimes \mathbb{N}_0}$ is uniquely defined by the \mathbb{C} -linear extension of the map

$$\mathbf{a} \otimes f_0 \otimes f_1 \otimes \cdots \mapsto C(\mathbf{a} \otimes \mathbb{1}_{\mathcal{L}}) \otimes f_0 \otimes f_1 \otimes \cdots$$

such that $\iota^* \alpha^n \iota = T^n$ for all $n \in \mathbb{N}_0$. We will draw on this approach for the concrete construction of noncommutative stationary Markov processes in Section 5.

Remark 3.4.4. Using the terminology of Kümmerer [57], the construction of the automorphism C in Example 3.4.3 provides us with a ‘tensor dilation of first order’. Such a ‘dilation of first order’ can always be further upgraded to a Markov dilation, as illustrated in Example 3.4.3. Note that Theorem 3.4.2, as a result on the existence of a Markov dilation, includes of course the existence of a ‘Markov dilation of first order’ which can be further amplified as done in Example 3.4.3. We will take this viewpoint in Theorem 5.2.3.

4. MARKOVIANITY FROM REPRESENTATIONS OF THE THOMPSON MONOID F^+

The noncommutative de Finetti theorem, Theorem 1.0.4, rests on the result that representations of the partial shift monoid S^+ on noncommutative probability spaces provide rich structures of commuting squares, in particular as they are underlying the notion of noncommutative Bernoulli shifts in Definition 2.6.15. Here we investigate commuting square structures as they emerge from representations of the Thompson monoid F^+ on noncommutative probability spaces. Our investigations reveal that certain commuting squares, as available in triangular towers of inclusions, already encode Markovianity. As the partial shift monoid S^+ is a quotient of the Thompson monoid F^+ , our approach yields de-Finetti-type results for noncommutative stationary Markov processes.

Let us fix some notation, as it will be used throughout this section. We assume that the probability space (\mathcal{M}, ψ) is equipped with the representation $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$. For brevity of notion, especially in proofs, the represented generators of F^+ are also denoted by

$$\alpha_n := \rho(g_n) \in \text{End}(\mathcal{M}, \psi),$$

with fixed point algebras given by $\mathcal{M}^{\alpha_n} := \{x \in \mathcal{M} \mid \alpha_n(x) = x\}$, for $0 \leq n < \infty$. Furthermore the intersections of fixed point algebras

$$\mathcal{M}_n := \bigcap_{k \geq n+1} \mathcal{M}^{\alpha_k}$$

give the tower of von Neumann subalgebras

$$\mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots \subset \mathcal{M}_\infty := \bigvee_{n \geq 0} \mathcal{M}_n \subset \mathcal{M}.$$

From the viewpoint of noncommutative probability theory, this tower provides a filtration of the noncommutative probability space (\mathcal{M}, ψ) . In particular, we will see in Subsection 4.2 that the inclusions

$$\begin{array}{ccc} \mathcal{M}_m & \subset & \mathcal{M} \\ \cup & & \cup \\ \alpha_0^m(\mathcal{M}_0) & \subset & \alpha_0^m(\mathcal{M}_\infty) \end{array}$$

form commuting squares which encode Markovianity. Consequently the canonical local filtration of a stationary process $(\mathcal{M}, \psi, \alpha_0, \mathcal{A}_0)$ will be seen to be a local subfiltration of a local Markov filtration whenever the ψ -conditioned von Neumann subalgebra \mathcal{A}_0 is well-localized, to be more precise: contained in the intersection of fixed point algebras \mathcal{M}_0 . It is worthwhile to emphasize that, depending on the choice of the generator \mathcal{A}_0 , the canonical local filtration of this stationary process may not be Markovian. Subsection 4.3 investigates in detail conditions under which the canonical local filtration of a stationary process $(\mathcal{M}, \psi, \alpha_0, \mathcal{A}_0)$ is Markovian. Finally, Subsection 4.4 provides the proof of Theorem 1.0.5, a noncommutative de Finetti theorem as appropriate for noncommutative stationary Markov processes.

4.1. Representations with a generating property. An immediate consequence of the relations between generators of the Thompson monoid F^+ is the adaptedness of the endomorphism α_0 to the tower of (intersected) fixed point algebras:

$$\alpha_0(\mathcal{M}_n) \subset \mathcal{M}_{n+1} \quad \text{for all } n \in \mathbb{N}_0.$$

Thus, generalizing terminology from classical probability, the random variables

$$\begin{aligned} \iota_0 &:= \text{Id}|_{\mathcal{M}_0} : \mathcal{M}_0 \rightarrow \mathcal{M}_0 \subset \mathcal{M} \\ \iota_1 &:= \alpha_0|_{\mathcal{M}_0} : \mathcal{M}_0 \rightarrow \mathcal{M}_1 \subset \mathcal{M} \\ \iota_2 &:= \alpha_0^2|_{\mathcal{M}_0} : \mathcal{M}_0 \rightarrow \mathcal{M}_2 \subset \mathcal{M} \\ &\vdots \\ \iota_n &:= \alpha_0^n|_{\mathcal{M}_0} : \mathcal{M}_0 \rightarrow \mathcal{M}_n \subset \mathcal{M} \end{aligned}$$

are adapted to the filtration $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$ and α_0 is the time evolution of the stationary process $(\mathcal{M}, \psi, \alpha_0, \mathcal{M}_0)$. We refer the reader to [39, Chapter 3] for more information on the general philosophy of adapted endomorphisms and to [40, Appendix A] or [31] on how adaptedness is of relevance within the context of distributional symmetries and invariance principles.

Clearly, at most, the von Neumann subalgebra \mathcal{M}_∞ can be generated by this sequence of random variables $(\iota_n)_{n \geq 0}$. An immediate question is whether a representation of the Thompson monoid F^+ restricts to the von Neumann subalgebra \mathcal{M}_∞ .

Definition 4.1.1. The representation $\rho : F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ is said to have the *generating property* if $\mathcal{M}_\infty = \mathcal{M}$.

As shown in Proposition 4.1.5 below, this generating property entails that each intersected fixed point algebra $\mathcal{M}_n = \bigcap_{k \geq n+1} \mathcal{M}^{\alpha_k}$ equals the single fixed point algebra $\mathcal{M}^{\alpha_{n+1}}$. Thus the generating property tremendously simplifies the form of the tower $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \dots$, and our next result shows that this can always be achieved by restriction.

Proposition 4.1.2. *The representation $\rho : F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ restricts to the generating representation $\rho_{\text{gen}} : F^+ \rightarrow \text{End}(\mathcal{M}_\infty, \psi_\infty)$ such that $\alpha_n(\mathcal{M}_\infty) \subset \mathcal{M}_\infty$ and $E_{\mathcal{M}_\infty} E_{\mathcal{M}^{\alpha_n}} = E_{\mathcal{M}^{\alpha_n}} E_{\mathcal{M}_\infty}$ for all $n \in \mathbb{N}_0$. Here ψ_∞ denotes the restriction of the state ψ to \mathcal{M}_∞ . $E_{\mathcal{M}^{\alpha_n}}$ and $E_{\mathcal{M}_\infty}$ denote the unique ψ -preserving normal conditional expectations onto \mathcal{M}^{α_n} and \mathcal{M}_∞ respectively.*

Proof. We show that $\alpha_i(\mathcal{M}_n) \subset \mathcal{M}_{n+1}$ for all $i, n \geq 0$. Let $x \in \mathcal{M}_n$. If $i \geq n+1$ then $\alpha_i(x) = x$ is immediate from the definition of \mathcal{M}_n . If $i < n+1$ then, using the relations for the generators of the Thompson monoid, $\alpha_i(x) = \alpha_i \alpha_{k+1}(x) = \alpha_{k+2} \alpha_i(x)$ for any $k \geq n$, thus $\alpha_i(x) \in \mathcal{M}_{n+1}$. Consequently α_i maps $\bigcup_{n \geq 0} \mathcal{M}_n$ into itself for any $i \in \mathbb{N}_0$. Now a standard approximation argument shows that \mathcal{M}_∞ is invariant under α_i for any $i \in \mathbb{N}_0$. Consequently the representation ρ restricts to \mathcal{M}_∞ and, of course, this restriction ρ_{gen} has the generating property.

Since \mathcal{M}_∞ is globally invariant under the modular automorphism group of (\mathcal{M}, ψ) , there exists the (unique) ψ -preserving normal conditional expectation $E_{\mathcal{M}_\infty}$ from \mathcal{M} onto \mathcal{M}_∞ . In particular, $\rho_{\text{gen}}(g_n) = \alpha_n|_{\mathcal{M}_\infty}$ commutes with the modular automorphism

group of $(\mathcal{M}_\infty, \psi_\infty)$ which ensures $\rho_{\text{gen}}(g_n) \in \text{End}(\mathcal{M}_\infty, \psi_\infty)$. Finally that $E_{\mathcal{M}_\infty}$ and $E_{\mathcal{M}^{\alpha_n}}$ commute is concluded from

$$E_{\mathcal{M}_\infty} \alpha_n E_{\mathcal{M}_\infty} = \alpha_n E_{\mathcal{M}_\infty},$$

which implies $E_{\mathcal{M}^{\alpha_n}} E_{\mathcal{M}_\infty} = E_{\mathcal{M}_\infty} E_{\mathcal{M}^{\alpha_n}}$ by routine arguments, and an application of the mean ergodic theorem (see for example [53, Theorem 8.3]),

$$E_{\mathcal{M}^{\alpha_n}} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \alpha_n^i,$$

where the limit is taken in the pointwise strong operator topology. \square

Lemma 4.1.3. *With the notations as above, $\mathcal{M}_k = \mathcal{M}^{\alpha_{k+1}} \cap \mathcal{M}_\infty$ for all $k \in \mathbb{N}_0$.*

Proof. For the sake of brevity of notation, let $Q_n = E_{\mathcal{M}^{\alpha_n}}$ denote the ψ -preserving normal conditional expectation from \mathcal{M} onto \mathcal{M}^{α_n} . By the definition of \mathcal{M}_k and \mathcal{M}_∞ , it is clear that $\mathcal{M}_k \subset \mathcal{M}^{\alpha_{k+1}} \cap \mathcal{M}_\infty$. In order to show the reverse inclusion, it suffices to show that $Q_n Q_k|_{\mathcal{M}_\infty} = Q_k|_{\mathcal{M}_\infty}$, $0 \leq k < n < \infty$. We claim that, for $0 \leq k < n$,

$$Q_n Q_k|_{\mathcal{M}_\infty} = Q_k|_{\mathcal{M}_\infty} \iff Q_k Q_n Q_k|_{\mathcal{M}_\infty} = Q_k|_{\mathcal{M}_\infty}.$$

Indeed this equivalence is immediate from

$$\begin{aligned} \psi((Q_n Q_k - Q_k)(y^*)(Q_n Q_k - Q_k)(x)) &= \psi(y^*(Q_k Q_n - Q_k)(Q_n Q_k - Q_k)(x)) \\ &= \psi(y^*(Q_k - Q_k Q_n Q_k)(x)) \end{aligned}$$

for all $x, y \in \mathcal{M}_\infty$. We are left to prove $Q_k Q_n Q_k|_{\mathcal{M}_\infty} = Q_k|_{\mathcal{M}_\infty}$ for $k < n$. For this purpose we express the conditional expectations Q_k and Q_n as mean ergodic limits in the pointwise strong operator topology and calculate

$$\begin{aligned} Q_k Q_n Q_k|_{\mathcal{M}_\infty} &= \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{MN} \sum_{i=1}^M \sum_{j=1}^N \alpha_k^i \alpha_n^j Q_k|_{\mathcal{M}_\infty} \\ &= \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{MN} \sum_{i=1}^M \sum_{j=1}^N \alpha_{n+i}^j \alpha_k^i Q_k|_{\mathcal{M}_\infty} \\ &= \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{MN} \sum_{i=1}^M \sum_{j=1}^N \alpha_{n+i}^j Q_k|_{\mathcal{M}_\infty} \\ &= \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{i=1}^M Q_{n+i} Q_k|_{\mathcal{M}_\infty} = Q_k|_{\mathcal{M}_\infty}. \end{aligned}$$

Here the last equality follows because for $x \in \mathcal{M}_\infty$, also $Q_k x \in \mathcal{M}_\infty$ and so it holds that $Q_{n+i} Q_k(x) = Q_k(x)$ for i sufficiently large, thus

$$\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{i=1}^M Q_{n+i} = \text{Id}$$

in the pointwise strong operator topology. \square

Corollary 4.1.4. *The following set of inclusions forms a commuting square for every $n \in \mathbb{N}_0$:*

$$\begin{array}{ccc} \mathcal{M}^{\alpha_{n+1}} & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{M}_n & \subset & \mathcal{M}_\infty \end{array}$$

Proof. Let Q_n and $E_{\mathcal{M}_\infty}$ be the ψ -preserving normal conditional expectation from \mathcal{M} onto \mathcal{M}^{α_n} and \mathcal{M}_∞ respectively for $n \in \mathbb{N}_0$. For $n \in \mathbb{N}_0$, by Proposition 4.1.2, $Q_{n+1}E_{\mathcal{M}_\infty} = E_{\mathcal{M}_\infty}Q_{n+1}$ and by Lemma 4.1.3, $\mathcal{M}_n = \mathcal{M}^{\alpha_{n+1}} \cap \mathcal{M}_\infty$. By (iv) of Proposition 2.5.1, we get a commuting square. \square

Proposition 4.1.5. *If the representation $\rho : F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ has the generating property then the following equality holds for all $n \in \mathbb{N}_0$:*

$$\mathcal{M}_n = \mathcal{M}^{\alpha_{n+1}}.$$

In other words, one has the tower of fixed point algebras

$$\mathcal{M}^{\rho(F^+)} \subset \mathcal{M}^{\rho(g_0)} \subset \mathcal{M}^{\rho(g_1)} \subset \mathcal{M}^{\rho(g_2)} \subset \dots \subset \mathcal{M} = \bigvee_{n \geq 0} \mathcal{M}^{\rho(g_n)}.$$

Proof. If the representation ρ is generating, then $\mathcal{M}_\infty = \mathcal{M}$. Hence $\mathcal{M}_n = \mathcal{M}^{\alpha_{n+1}}$ for all $n \in \mathbb{N}_0$ as a consequence of Lemma 4.1.3. \square

Remark 4.1.6. Suppose that the representation $\rho : F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ satisfies the additional relations $\rho(g_n)\rho(g_n) = \rho(g_{n+1})\rho(g_n)$ for all $n \in \mathbb{N}_0$, as it is the case for representations of the partial shifts monoid S^+ . Then the inclusions $\mathcal{M}^{\rho(g_n)} \subset \mathcal{M}^{\rho(g_{n+1})}$, and consequently $\mathcal{M}_n = \mathcal{M}^{\rho(g_{n+1})}$, are immediate without stipulating the generating property of the representation, since $x = \rho(g_n)(x)$ implies $x = \rho(g_n^2)(x) = \rho(g_{n+1})\rho(g_n)(x) = \rho(g_{n+1})(x)$ for all $x \in \mathcal{M}$ and $n \in \mathbb{N}_0$.

4.2. Commuting squares and Markovianity for shifted fixed point algebras. The following intertwining properties will be crucial for obtaining local Markov filtrations from representations of the Thompson monoid F^+ .

Proposition 4.2.1. *Suppose $\rho : F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ is a (not necessarily generating) representation of F^+ . Then with $\alpha_n = \rho(g_n)$, the following equality holds:*

$$\alpha_k Q_n = Q_{n+1} \alpha_k$$

for all $0 \leq k < n < \infty$. Here Q_n denotes the ψ -preserving normal conditional expectation from \mathcal{M} onto the fixed point algebra \mathcal{M}^{α_n} of the represented generator $\alpha_n \in \text{End}(\mathcal{M}, \psi)$.

Proof. An application of the mean ergodic theorem and the relations between the generators of the Thompson monoid F^+ yield that, for $k < n$,

$$\alpha_k Q_n = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^{N-1} \alpha_k \alpha_n^i = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} \alpha_{n+1}^i \alpha_k = Q_{n+1} \alpha_k.$$

Here the limits are taken in the pointwise strong operator topology. \square

Theorem 4.2.2. *Suppose $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ is a generating representation with $\alpha_k := \rho(g_k)$ for all $k \in \mathbb{N}_0$. Then each cell in the following triangular tower is a commuting square:*

$$\begin{array}{cccccccc}
 \mathcal{M}_0 & \subset & \mathcal{M}_1 & \subset & \mathcal{M}_2 & \subset & \mathcal{M}_3 & \subset \cdots \subset \mathcal{M}_\infty = \mathcal{M} \\
 & & \cup & & \cup & & \cup & \cup \\
 & & \alpha_0(\mathcal{M}_0) & \subset & \alpha_0(\mathcal{M}_1) & \subset & \alpha_0(\mathcal{M}_2) & \subset \cdots \subset \alpha_0(\mathcal{M}_\infty) \\
 & & & & \cup & & \cup & \cup \\
 & & & & \alpha_0^2(\mathcal{M}_0) & \subset & \alpha_0^2(\mathcal{M}_1) & \subset \cdots \subset \alpha_0^2(\mathcal{M}_\infty) \\
 & & & & & & \cup & \cup \\
 & & & & & & \vdots & \vdots \\
 & & & & & & \vdots & \vdots
 \end{array}$$

In particular, $\mathcal{M}_{n+1} \cap \alpha_0(\mathcal{M}_{n+1}) = \alpha_0(\mathcal{M}_n)$ for all $n \geq 0$.

Proof. Let $0 \leq m < n < \infty$ and $k \geq 1$. We verify first all inclusions as they appear in the diagram

$$\begin{array}{ccc}
 \mathcal{M}_{m+k} & \subset & \mathcal{M}_{n+k} \\
 \cup & & \cup \\
 \alpha_0^k(\mathcal{M}_m) & \subset & \alpha_0^k(\mathcal{M}_n)
 \end{array} \quad (4.2.1)$$

Indeed, the definition of \mathcal{M}_n ensures the claimed horizontal inclusions in this diagram. The vertical inclusions in the diagram follow from the intertwining properties $\alpha_0^k Q_{n+1} = Q_{n+1+k} \alpha_0^k$ (see Proposition 4.2.1). For $n = \infty$, all inclusions are easily concluded by routine approximation arguments.

We show next that the above diagram is a commuting square. Indeed, as ρ is generating, $\mathcal{M}_\ell = \mathcal{M}^{\alpha_{\ell+1}}$, for all $\ell \in \mathbb{N}_0$, and $E_{\mathcal{M}_\ell} = Q_{\ell+1}$, where $E_{\mathcal{M}_\ell}$ is the conditional expectation onto \mathcal{M}_ℓ . Hence for any $x \in \mathcal{M}_n$,

$$E_{\mathcal{M}_{m+k}} \alpha_0^k(x) = Q_{m+k+1} \alpha_0^k(x) = \alpha_0^k Q_{m+1}(x) = \alpha_0^k E_{\mathcal{M}_m}(x).$$

This ensures that $E_{\mathcal{M}_{m+k}}(\alpha_0^k(\mathcal{M}_n)) = \alpha_0^k(\mathcal{M}_m)$. Thus the above inclusions form a commuting square by Proposition 2.5.1 and, in particular, it holds that $\alpha_0^k(\mathcal{M}_m) = \mathcal{M}_{m+k} \cap \alpha_0^k(\mathcal{M}_n)$.

Finally, the commuting square properties of more general cells in the triangular tower of inclusions are deduced from those in (4.2.1), since commuting square properties are preserved when acting with the endomorphism α_0 on all four corners of the diagram. \square

Corollary 4.2.3. *Suppose $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ is a generating representation with $\alpha_k := \rho(g_k)$ for all $k \in \mathbb{N}_0$. Let $0 \leq m \leq n < \infty$ be fixed. Then each cell in the following triangular tower is a commuting square:*

$$\begin{array}{cccccccc}
 \mathcal{M}_m & \subset & \mathcal{M}_{n+1} & \subset & \mathcal{M}_{n+2} & \subset & \cdots & \subset \mathcal{M}_\infty = \mathcal{M} \\
 & & \cup & & \cup & & & \cup \\
 & & \alpha_m(\mathcal{M}_m) & \subset & \alpha_m(\mathcal{M}_{n+1}) & \subset & \cdots & \subset \alpha_m(\mathcal{M}_\infty) \\
 & & & & \cup & & & \cup \\
 & & & & \alpha_m^2(\mathcal{M}_m) & \subset & \cdots & \subset \alpha_m^2(\mathcal{M}_\infty) \\
 & & & & & & & \cup \\
 & & & & & & \vdots & \vdots
 \end{array}$$

In particular, $\mathcal{M}_{n+1} \cap \alpha_m(\mathcal{M}_{n+1}) = \alpha_m(\mathcal{M}_m)$ and $\mathcal{M}_{n+k+1} \cap \alpha_m(\mathcal{M}_{n+k+1}) = \alpha_m(\mathcal{M}_{n+k})$ for all $k \geq 1$.

Proof. Consider the representation $\rho_{m,n} := \rho \circ \text{sh}_{m,n}: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ where $\text{sh}_{m,n}$ denotes the (m, n) -partial shift as introduced in Definition 2.1.1. We observe that $\rho_{m,n}(g_0) = \rho(g_m)$ and $\rho_{m,n}(g_k) = \rho(g_{n+k})$ for all $k \geq 1$. In particular this ensures that $\rho_{m,n}$ inherits the generating property from the representation ρ . Thus Theorem 4.2.2 applies to $\rho_{m+1, n+1}$ and all claimed properties are immediate since $\mathcal{M}_m = \mathcal{M}^{\rho(g_{m+1})} = \mathcal{M}^{\rho_{m+1, n+1}(g_0)}$ and $\mathcal{M}_{n+k} = \mathcal{M}^{\rho(g_{n+k+1})} = \mathcal{M}^{\rho_{m+1, n+1}(g_k)}$ for $k \geq 1$. \square

The triangular tower of α_0 -shifted fixed point algebras (as given in Theorem 4.2.2) can also be addressed through a local filtration indexed by ‘intervals’. This reveals that Markovianity (as introduced in Definition 2.5.7) corresponds to specific commuting squares in the triangular tower.

Corollary 4.2.4. *Suppose $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ is a generating representation. The family of von Neumann subalgebras $\mathcal{M}_\bullet^\rho \equiv \{\mathcal{M}_I^\rho\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ of (\mathcal{M}, ψ) , with*

$$\mathcal{M}_{[0, n]}^\rho := \mathcal{M}_n, \quad \mathcal{M}_{[m, m+n]}^\rho := \rho(g_0^m)(\mathcal{M}_n), \quad \mathcal{M}_{[m, \infty]}^\rho := \rho(g_0^m)(\mathcal{M}_\infty),$$

defines a local Markov filtration.

Proof. First we check the isotony property to verify that this family of subalgebras forms a local filtration. Let $\alpha_n = \rho(g_n), n \in \mathbb{N}_0$, as before. Suppose $[m, m+n] \subset [k, k+\ell]$, we will show that $\mathcal{M}_{[m, m+n]}^\rho \subset \mathcal{M}_{[k, k+\ell]}^\rho$, that is, $\alpha_0^m(\mathcal{M}_n) \subset \alpha_0^k(\mathcal{M}_\ell)$. As $[m, m+n] \subset [k, k+\ell]$, we must have $m \geq k$ and $n \leq \ell$. Hence for $x \in \mathcal{M}_n$, we can write $\alpha_0^m(x) = \alpha_0^k \alpha_0^{m-k}(x)$, so it suffices to show that $\alpha_0^{m-k}(x) \in \mathcal{M}_\ell$. Let $p \geq \ell + 1 \geq (m-k) + n + 1$, then $\alpha_p \alpha_0^{m-k}(x) = \alpha_0^{m-k} \alpha_{p-(m-k)}(x) = \alpha_0^{m-k}(x)$ as $p - (m-k) \geq n + 1$.

Let P_I^ρ denote the ψ -preserving normal conditional expectation from \mathcal{M} onto \mathcal{M}_I^ρ . This local filtration is Markovian if $P_{[0, m]}^\rho P_{[m, n]}^\rho = P_{[m, m]}^\rho$ for $0 \leq m \leq n$, which is implied by the definition of \mathcal{M}_I^ρ and the following cell of inclusions that is a commuting square as a consequence of Theorem 4.2.2:

$$\begin{array}{ccc} \mathcal{M}_m & \subset & \mathcal{M} \\ \cup & & \cup \\ \alpha_0^m(\mathcal{M}_0) & \subset & \alpha_0^m(\mathcal{M}_n) \end{array}.$$

\square

Corollary 4.2.5. *Suppose $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ is a generating representation and consider the (m, n) -shifted representation $\rho_{m,n} := \rho \circ \text{sh}_{m,n}$ for some fixed $0 \leq m \leq n < \infty$. Then the family of von Neumann subalgebras $\mathcal{M}_\bullet^{\rho_{m,n}} \equiv \{\mathcal{M}_I^{\rho_{m,n}}\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ of (\mathcal{M}, ψ) , with*

$$\mathcal{M}_{[0, \ell]}^{\rho_{m,n}} := \mathcal{M}_{\ell+n}, \quad \mathcal{M}_{[k, k+\ell]}^{\rho_{m,n}} := \rho(g_m^k)(\mathcal{M}_{\ell+n}), \quad \mathcal{M}_{[k, \infty]}^{\rho_{m,n}} := \rho(g_m^k)(\mathcal{M}_\infty),$$

defines a local Markov filtration.

Proof. The case $m = n = 0$ corresponds to Corollary 4.2.4. Its proof directly transfers to the general case $0 \leq m \leq n$, after relabeling the involved objects and morphisms according to the (m, n) -shifted representation. \square

4.3. Commuting squares and Markovianity for stationary processes. Given the representation $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$, with represented generators $\alpha_n := \rho(g_n)$, for $n \in \mathbb{N}_0$, and intersected fixed point algebras

$$\mathcal{M}_n := \bigcap_{k \geq n+1} \mathcal{M}^{\alpha_k},$$

let $\mathcal{A}_0 \subset \mathcal{M}_0$ be a von Neumann subalgebra of (\mathcal{M}, ψ) . Then $(\mathcal{M}, \psi, \alpha_0, \mathcal{A}_0)$ is a (unilateral noncommutative) stationary process with generating algebra \mathcal{A}_0 . Its canonical local filtration is denoted by $\mathcal{A}_\bullet \equiv \{\mathcal{A}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)}$, where

$$\mathcal{A}_I := \bigvee_{i \in I} \alpha_0^i(\mathcal{A}_0),$$

and an ‘interval’ $I \in \mathcal{I}(\mathbb{N}_0)$ is written as $[m, n] := \{i \in \mathbb{N}_0 \mid m \leq i \leq n\}$ or $[m, \infty) := \{i \in \mathbb{N}_0 \mid m \leq i\}$.

Furthermore P_I will denote the ψ -preserving normal conditional expectation from \mathcal{M} onto \mathcal{A}_I . Note that the endomorphism α_0 acts covariantly on the local filtration, i.e. $\alpha_0(\mathcal{A}_I) = \mathcal{A}_{I+1}$ for all $I \in \mathcal{I}(\mathbb{N}_0)$, where $I+1 := \{i+1 \mid i \in I\}$.

We record a simple, but important, observation obtained from the relations of F^+ on stationary processes to which we will frequently appeal.

Proposition 4.3.1. *Let $(\mathcal{M}, \psi, \alpha_0, \mathcal{A}_0)$ be the (unilateral noncommutative) stationary process with $\mathcal{A}_0 \subset \mathcal{M}_0$ as above. Then it holds that $\mathcal{A}_{[0,n]} \subset \mathcal{M}_n$ for all $n \in \mathbb{N}_0$.*

Proof. As $\mathcal{A}_0 \subset \mathcal{M}_0$, it holds that $\alpha_n(x) = x$ for any $x \in \mathcal{A}_0$ and $n \in \mathbb{N}$. Thus using the defining relations of F^+ we get for $0 \leq k \leq n$ and $n+1 \leq l$,

$$\alpha_l \alpha_0^k(x) = \alpha_0^k \alpha_{l-k}(x) = \alpha_0^k(x).$$

Hence $\mathcal{A}_{[0,n]} \subset \mathcal{M}_n$ for all $n \in \mathbb{N}_0$. □

Remark 4.3.2. The canonical local filtration \mathcal{A}_\bullet is coarser than the local filtration $\mathcal{M}_\bullet^\rho \equiv \{\mathcal{M}_I^\rho\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ which is Markovian whenever ρ is generating by Corollary 4.2.4. Indeed this follows from the endomorphism α_0 acting covariantly on the local filtration. More explicitly, we conclude that $\mathcal{A}_{[m,n]} = \alpha_0^m(\mathcal{A}_{[0,n-m]}) \subset \alpha_0^m(\mathcal{M}_{n-m}) = \mathcal{M}_{[m,n]}^\rho$ for $0 \leq m \leq n$. Furthermore it holds that

$$\mathcal{A}_{[m,\infty)} = \bigvee_{n \geq m} \mathcal{A}_{[m,n]} \subset \bigvee_{n \geq m} \mathcal{M}_{[m,n]}^\rho = \mathcal{M}_{[m,\infty)}^\rho.$$

We next observe that the generating property of the representation ρ can be concluded from the minimality of a stationary process.

Proposition 4.3.3. *Suppose the representation $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ and $\mathcal{A}_0 \subset \mathcal{M}_0$ are given. If the stationary process $(\mathcal{M}, \psi, \alpha_0, \mathcal{A}_0)$ is minimal, then ρ is generating.*

Proof. The minimality of the stationary process $(\mathcal{M}, \psi, \alpha_0, \mathcal{A}_0)$ ensures $\mathcal{A}_{[0,\infty)} = \bigvee_{i \in \mathbb{N}_0} \alpha_0^i(\mathcal{A}_0) = \mathcal{M}$. By Proposition 4.3.1, $\mathcal{A}_{[0,n]} \subset \mathcal{M}_n$ for all $n \in \mathbb{N}_0$. Thus $\mathcal{M} = \bigvee_{n \geq 0} \mathcal{A}_{[0,n]} \subset \bigvee_{n \geq 0} \mathcal{M}_n = \mathcal{M}_\infty$. We conclude from this that the representation ρ has the generating property, i.e. $\mathcal{M}_\infty = \mathcal{M}$. □

In the following results, it is not assumed that the stationary process is minimal or that the representation ρ is generating unless explicitly mentioned.

Theorem 4.3.4. *Suppose $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ is a representation. Let $\alpha_n := \rho(g_n)$ as before, and let $\mathcal{A}_0 \subset \mathcal{M}_0$ and $\mathcal{A}_{[0,\infty)} := \bigvee_{n \in \mathbb{N}_0} \alpha_0^n(\mathcal{A}_0)$ be von Neumann subalgebras of (\mathcal{M}, ψ) such that the inclusions*

$$\begin{array}{ccc} \mathcal{M}^{\alpha_1} & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{A}_0 & \subset & \mathcal{A}_{[0,\infty)} \end{array}$$

form a commuting square. Then the family of von Neumann subalgebras $\mathcal{A}_\bullet \equiv \{\mathcal{A}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)}$, with

$$\mathcal{A}_I := \bigvee_{i \in I} \alpha_0^i(\mathcal{A}_0)$$

is a local Markov filtration and $(\mathcal{M}, \psi, \alpha_0, \mathcal{A}_0)$ is a stationary Markov process.

Proof. Let Q_n and P_I denote the ψ -preserving normal conditional expectations from \mathcal{M} onto \mathcal{M}^{α_n} and \mathcal{A}_I respectively as before. Note that the commuting square condition implies $Q_1 P_{[0,\infty)} = P_{[0,0]}$. From Proposition 4.3.1, $\mathcal{A}_{[0,n]} \subset \mathcal{M}_n \subset \mathcal{M}^{\alpha_{n+1}}$ for all $n \in \mathbb{N}_0$. Hence we get

$$\begin{aligned} P_{[0,n]} \alpha_0^n P_{[0,\infty)} &= P_{[0,n]} Q_{n+1} \alpha_0^n P_{[0,\infty)} && \text{(since } \mathcal{A}_{[0,n]} \subset \mathcal{M}^{\alpha_{n+1}}) \\ &= P_{[0,n]} \alpha_0^n Q_1 P_{[0,\infty)} && \text{(by intertwining property)} \\ &= P_{[0,n]} \alpha_0^n P_{[0,0]} P_{[0,\infty)} && \text{(by commuting square condition)} \\ &= \alpha_0^n P_{[0,0]} P_{[0,\infty)} && \text{(as } \mathcal{A}_{[n,n]} \subset \mathcal{A}_{[0,n]}) \\ &= P_{[n,n]} \alpha_0^n P_{[0,0]} P_{[0,\infty)} && \text{(since } \mathcal{A}_{[n,n]} = \alpha_0^n(\mathcal{A}_0)) \\ &= P_{[n,n]} \alpha_0^n Q_1 P_{[0,\infty)} && \text{(by commuting square condition)} \\ &= P_{[n,n]} Q_{n+1} \alpha_0^n P_{[0,\infty)} && \text{(by intertwining property)} \\ &= P_{[n,n]} \alpha_0^n P_{[0,\infty)} && \text{(since } \mathcal{A}_{[n,n]} \subset \mathcal{M}^{\alpha_{n+1}}). \end{aligned}$$

Altogether we have shown that $P_{[0,n]} P_{[n,\infty)} = P_{[n,n]}$, which is the required Markovianity for the local filtration $\{\mathcal{A}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)}$. \square

Corollary 4.3.5. *Suppose $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ is a representation with $\alpha_0 = \rho(g_0)$. Then the quadruple $(\mathcal{M}, \psi, \alpha_0, \mathcal{M}_0)$ is a stationary Markov process.*

Proof. We know from Corollary 4.1.4 that the following is a commuting square:

$$\begin{array}{ccc} \mathcal{M}^{\alpha_1} & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{M}_0 & \subset & \mathcal{M}_\infty \end{array}$$

Let $\{\mathcal{M}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ denote the local filtration given by $\mathcal{M}_I = \bigvee_{i \in I} \alpha_0^i(\mathcal{M}_0)$ and P_I be the corresponding conditional expectations. As $\mathcal{M}_{[0,n]} \subset \mathcal{M}_n$ for all $n \in \mathbb{N}_0$, it is easily verified that $\mathcal{M}_{[0,\infty)} \subset \mathcal{M}_\infty$. Let $P_0 := P_{[0,0]}$ be the ψ -preserving conditional expectation from \mathcal{M} onto \mathcal{M}_0 . Then from the commuting square above, we have $E_{\mathcal{M}_\infty} Q_1 = P_0$, where $E_{\mathcal{M}_\infty}$ is of course the conditional expectation onto \mathcal{M}_∞ . This in

turn gives $P_{[0,\infty)}Q_1 = P_{[0,\infty)}E_{\mathcal{M}_\infty}Q_1 = P_{[0,\infty)}P_0 = P_0$. Hence we get that \mathcal{M}_0 is a von Neumann subalgebra of \mathcal{M} such that

$$\begin{array}{ccc} \mathcal{M}^{\alpha_1} & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{M}_0 & \subset & \mathcal{M}_{[0,\infty)} \end{array}$$

forms a commuting square. By Theorem 4.3.4, $(\mathcal{M}, \psi, \alpha_0, \mathcal{M}_0)$ is a stationary Markov process. \square

Corollary 4.3.6. *Suppose $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ is a representation with $\alpha_m = \rho(g_m)$, for $m \in \mathbb{N}_0$. Then the quadruple $(\mathcal{M}, \psi, \alpha_m, \mathcal{M}_n)$ is a stationary Markov process for any $0 \leq m \leq n < \infty$.*

Proof. Consider the representation $\rho_{m,n} := \rho \circ \text{sh}_{m,n}: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ where $\text{sh}_{m,n}$ denotes the (m, n) -partial shift as introduced in Definition 2.1.1. We observe that $\rho_{m,n}(g_0) = \rho(g_m)$ and $\rho_{m,n}(g_k) = \rho(g_{n+k})$ for all $k \geq 1$. In particular we get $\bigcap_{k \geq 1} \mathcal{M}^{\rho_{m,n}(g_k)} = \bigcap_{k \geq 1} \mathcal{M}^{\rho(g_{k+n})} = \bigcap_{k \geq n+1} \mathcal{M}^{\rho(g_k)} = \mathcal{M}_n$. Thus Corollary 4.3.5 applies for the (m, n) -shifted representation $\rho_{m,n}$ and its application completes the proof. \square

Corollary 4.3.7. *Suppose $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ is a generating representation. Then the quadruple $(\mathcal{M}, \psi, \alpha_m, \mathcal{M}^{\alpha_{n+1}})$ is a stationary Markov process for any $0 \leq m \leq n < \infty$.*

Proof. If the representation ρ is generating, then $\mathcal{M}^{\alpha_{n+1}} = \mathcal{M}_n$. Hence the result follows by Corollary 4.3.6. \square

Remark 4.3.8. The commuting square assumption in Theorem 4.3.4 may not be satisfied for a noncommutative stationary process $(\mathcal{M}, \psi, \alpha_0, \mathcal{A}_0)$ if one only demands that the generator \mathcal{A}_0 is a ψ -conditioned von Neumann subalgebra of the fixed point algebra \mathcal{M}^{α_1} . Consequently the canonical local filtration of the resulting noncommutative stationary processes may not be Markovian.

Theorem 4.3.9. *Let the probability space (\mathcal{M}, ψ) be equipped with the representation $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ and the local filtration $\mathcal{A}_\bullet \equiv \{\mathcal{A}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)}$, where $\mathcal{A}_I := \bigvee_{i \in I} \rho(g_0^i)(\mathcal{A}_0)$ for some von Neumann subalgebra \mathcal{A}_0 of \mathcal{M}_0 . Further suppose the inclusions*

$$\begin{array}{ccc} \mathcal{M}^{\rho(g_{m+1})} & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{A}_{[0,m]} & \subset & \mathcal{A}_{[0,\infty)} \end{array}$$

form a commuting square for all $m \geq 0$. Then each cell in the following triangular tower of inclusions is a commuting square:

$$\begin{array}{cccccccc} \mathcal{A}_{[0,0]} & \subset & \mathcal{A}_{[0,1]} & \subset & \mathcal{A}_{[0,2]} & \subset & \mathcal{A}_{[0,3]} & \subset & \mathcal{A}_{[0,4]} & \subset & \cdots & \subset & \mathcal{A}_{[0,\infty)} \\ & & \cup & & \cup & & \cup & & \cup & & & & \cup \\ & & \mathcal{A}_{[1,1]} & \subset & \mathcal{A}_{[1,2]} & \subset & \mathcal{A}_{[1,3]} & \subset & \mathcal{A}_{[1,4]} & \subset & \cdots & \subset & \mathcal{A}_{[1,\infty)} \\ & & & & \cup & & \cup & & \cup & & & & \cup \\ & & & & \mathcal{A}_{[2,2]} & \subset & \mathcal{A}_{[2,3]} & \subset & \mathcal{A}_{[2,4]} & \subset & \cdots & \subset & \mathcal{A}_{[2,\infty)} \\ & & & & & & \cup & & \cup & & & & \cup \\ & & & & & & \vdots & & \vdots & & & & \vdots \end{array}$$

In particular, \mathcal{A}_\bullet is a local Markov filtration.

Proof. All claimed inclusions in the triangular tower are clear from the definition of $\mathcal{A}_{[m,n]}$. We recall from Proposition 4.3.1 that $\alpha_0^k(\mathcal{A}_0) \subset \mathcal{M}^{\alpha_{n+1}}$ for $0 \leq k \leq n$. Hence $\mathcal{A}_{[m,n]} \subset \mathcal{M}^{\alpha_{n+1}}$ for all $0 \leq m \leq n$. Next we show that, for $0 \leq k$ and $1 \leq m$, the cell of inclusions

$$\begin{array}{ccc} \alpha_0^k(\mathcal{A}_{[0,m]}) & \subset & \alpha_0^k(\mathcal{A}_{[0,m+1]}) \\ \cup & & \cup \\ \alpha_0^{k+1}(\mathcal{A}_{[0,m-1]}) & \subset & \alpha_0^{k+1}(\mathcal{A}_{[0,m]}) \end{array}$$

forms a commuting square. So, as P_I denotes the normal ψ -preserving conditional expectation from \mathcal{M} onto \mathcal{A}_I , we need to show

$$P_{[k,m+k]}P_{[k+1,m+k+1]} = P_{[k+1,m+k]}$$

or, equivalently,

$$P_{[k,m+k]}\alpha_0^{k+1}P_{[0,m]} = \alpha_0^{k+1}P_{[0,m-1]}.$$

Indeed, we calculate

$$\begin{aligned} P_{[k,m+k]}\alpha_0^{k+1}P_{[0,m]} &= P_{[k,m+k]}Q_{m+k+1}\alpha_0^{k+1}P_{[0,m]} \\ &= P_{[k,m+k]}\alpha_0^{k+1}Q_mP_{[0,m]} \\ &= P_{[k,m+k]}\alpha_0^{k+1}Q_mP_{[0,\infty)}P_{[0,m]} \\ &= P_{[k,m+k]}\alpha_0^{k+1}P_{[0,m-1]}P_{[0,m]} \\ &= P_{[k,m+k]}\alpha_0^{k+1}P_{[0,m-1]} \\ &= \alpha_0^{k+1}P_{[0,m-1]}. \end{aligned}$$

Here we have used that $P_{[k,m+k]} = P_{[k,m+k]}Q_{m+k+1}$, the intertwining properties of α_0 and the commuting square assumption $Q_mP_{[0,\infty)} = P_{[0,m-1]}$.

Since $\alpha_0^k(\mathcal{A}_{[m,n]}) = \mathcal{A}_{[m+k,n+k]}$ is evident from the definition of the local filtration, we have verified that each cell of inclusions in the triangular tower forms a commuting square. \square

More generally, we may consider a probability space which is equipped both with a local filtration and a representation of the Thompson monoid, and formulate compatibility conditions between the local filtration and the representation such that one obtains rich commuting square structures.

Corollary 4.3.10. *Suppose the probability space (\mathcal{M}, ψ) is equipped with a local filtration $\mathcal{N}_\bullet \equiv \{\mathcal{N}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ and a representation $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ such that*

- (i) $\rho(g_0)(\mathcal{N}_I) = \mathcal{N}_{I+1}$ for all $I \in \mathcal{I}(\mathbb{N}_0)$ (compatibility),
- (ii) $\mathcal{N}_{[0,m]} \subset \mathcal{M}^{\rho(g_{m+1})}$ for all $m \in \mathbb{N}_0$ (adaptedness),
- (iii) the inclusions

$$\begin{array}{ccc} \mathcal{M}^{\rho(g_{m+1})} & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{N}_{[0,m]} & \subset & \mathcal{N}_{[0,\infty)} \end{array}$$

form a commuting square for all $m \in \mathbb{N}_0$.

Then each cell in the following triangular tower of inclusions is a commuting square:

$$\begin{array}{ccccccc}
 \mathcal{N}_{[0,0]} & \subset & \mathcal{N}_{[0,1]} & \subset & \mathcal{N}_{[0,2]} & \subset & \cdots & \subset & \mathcal{N}_{[0,\infty)} \\
 & & \cup & & \cup & & & & \cup \\
 & & \rho(g_0)(\mathcal{N}_{[0,0]}) & \subset & \rho(g_0)(\mathcal{N}_{[0,1]}) & \subset & \cdots & \subset & \rho(g_0)(\mathcal{N}_{[0,\infty)}) \\
 & & & & \cup & & & & \cup \\
 & & & & \rho(g_0^2)(\mathcal{N}_{[0,0]}) & \subset & \cdots & \subset & \rho(g_0^2)(\mathcal{N}_{[0,\infty)}) \\
 & & & & & & & & \cup \\
 & & & & & & & & \vdots \\
 & & & & & & & & \vdots
 \end{array}$$

In particular, \mathcal{N}_\bullet is a local Markov filtration.

Proof. Let P_I be the normal ψ -preserving conditional expectation onto \mathcal{N}_I . Let $\alpha_n = \rho(g_n)$ and Q_n be the normal ψ -preserving conditional expectation onto \mathcal{M}^{α_n} as before. We observe that $\mathcal{N} = \mathcal{N}_{[0,0]} \subset \mathcal{M}^{\alpha_1}$ by the given adaptedness. Adaptedness also gives us $\mathcal{N}_{[m,n]} \subset \mathcal{N}_{[0,n]} \subset \mathcal{M}^{\alpha_{n+1}}$ for $0 \leq m \leq n$. Thus $P_{[k,m+k]} = P_{[k,m+k]}Q_{m+k+1}$ as before. The rest of the proof follows just as in Theorem 4.3.9. \square

4.4. A noncommutative version of de Finetti's theorem. Most results of the previous two subsections can be reformulated in terms of sequences of random variables associated to stationary processes (see Definition 2.6.1).

Proposition 4.4.1. *Given the representation $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$, let \mathcal{A}_0 be some fixed ψ -conditioned von Neumann subalgebra of $\mathcal{M}_0 = \bigcap_{k \geq 1} \mathcal{M}^{\rho(g_k)}$ and $\varphi_0 := \psi|_{\mathcal{A}_0}$. Then the sequence of random variables*

$$(\iota_n)_{n \geq 0}: (\mathcal{A}_0, \varphi_0) \rightarrow (\mathcal{M}, \psi), \quad \iota_n := \rho(g_0)^n|_{\mathcal{A}_0}$$

(associated to the stationary process $(\mathcal{M}, \psi, \rho(g_0), \mathcal{A}_0)$) is partially spreadable. Furthermore this stationary process and its associated sequence of random variables have the same canonical local filtration

$$\mathcal{A}_\bullet \equiv \left\{ \mathcal{A}_I := \bigvee_{i \in I} \rho(g_0^i)(\mathcal{A}_0) \right\}_{I \in \mathcal{I}(\mathbb{N}_0)}$$

which is coarser than the local Markov filtration

$$\mathcal{M}_\bullet \equiv \left\{ \mathcal{M}_I := \bigvee_{i \in I} \rho(g_0^i)(\mathcal{M}_0) \right\}_{I \in \mathcal{I}(\mathbb{N}_0)}.$$

Proof. This is immediate from Definition 1.0.3, where we introduced partial spreadability as a distributional symmetry. Clearly the canonical filtration of the stationary process and its associated sequence of random variables coincide. The inclusion $\mathcal{A}_0 \subset \mathcal{M}_0$ ensures that \mathcal{A}_\bullet is coarser than \mathcal{M}_\bullet . The Markovianity of \mathcal{M}_\bullet is inferred from Corollary 4.3.5. \square

We are ready for the proof of a noncommutative version of de Finetti's theorem, as formulated in Theorem 1.0.5, and repeat its formulation for the convenience of the reader.

Theorem. *Let $\iota \equiv (\iota_n)_{n \geq 0}: (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ be a sequence of (identically distributed) random variables and consider the following conditions:*

- (a) ι is partially spreadable;
- (b) ι is stationary and adapted to a local Markov filtration;
- (c) ι is identically distributed and adapted to a local Markov filtration.

Then one has the following implications:

$$(a) \implies (b) \implies (c).$$

Proof (of Theorem 1.0.5). (a) \implies (b): The stationarity of ι follows from $\psi \circ \rho(g_0) = \psi$ and $\iota_n = \rho(g_0)^n \iota_0, n \geq 0$. Proposition 4.4.1 ensures that the canonical local filtration \mathcal{A}_\bullet is coarser than the local Markov filtration \mathcal{M}_\bullet . Thus the sequence ι is adapted to the local Markov filtration \mathcal{M}_\bullet , according to Definition 2.5.10.

(b) \implies (c): Stationary sequences are identically distributed. \square

Let (\mathcal{M}, ψ) be a noncommutative probability space, $\rho : F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ be a representation and \mathcal{A}_0 be a ψ -conditioned subalgebra of $\mathcal{M}_0 = \bigcap_{k \geq 1} \mathcal{M}^{\rho(g_k)}$. By Corollary 4.3.5, we know that if $\mathcal{A}_0 = \mathcal{M}_0$, then the canonical local filtration $\mathcal{A}_\bullet = \{\mathcal{A}_I\}_{I \in \mathcal{I}(\mathbb{N}_0)}$ is Markovian. The definition of a partially spreadable sequence $\{\iota_n\} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ entails the existence of a representation $\rho : F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ such that $\iota_0(\mathcal{A}) \subset \mathcal{M}_0 = \bigcap_{k \geq 1} \mathcal{M}^{\rho(g_k)}$ (see Definition 1.0.3). This motivates to strengthen the property of partial spreadability as follows.

Definition 4.4.2. A partially spreadable sequence of random variables $\iota \equiv (\iota_n)_{n \geq 0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ is said to be *maximal partially spreadable* if $\iota_0(\mathcal{A}) = \mathcal{M}_0 = \bigcap_{k \geq 1} \mathcal{M}^{\rho(g_k)}$. Here ρ denotes a representation of F^+ as in Definition 1.0.3, and \mathcal{M}_0 the intersection of the fixed point algebras $\mathcal{M}^{\rho(g_k)}$ for $k \geq 1$.

This refined notion of partial spreadability allows us to tightly connect the representation theory of the Thompson monoid F^+ and Markovianity.

Theorem 4.4.3. *A maximal partially spreadable sequence of random variables $\iota \equiv (\iota_n)_{n \geq 0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ is stationary and Markovian.*

Thus we have obtained a noncommutative de Finetti theorem for noncommutative stationary Markov sequences. One should not expect the converse to be true in the full generality of our operator algebraic framework, for similar reasons as outlined for the noncommutative extended de Finetti theorem in [53].

5. CONSTRUCTIONS OF REPRESENTATIONS OF THE THOMPSON MONOID F^+

This section is about how to construct representations of the Thompson monoid F^+ as they naturally arise in noncommutative probability theory. It will be seen that such constructions are intimately related to the construction of stationary Markov processes. In particular, this will establish that a large class of stationary Markov sequences is partially spreadable.

5.1. Tensor product constructions. Let (\mathcal{A}, φ) and (\mathcal{C}, χ) be probability spaces. Taking the infinite von Neumann algebraic tensor product with respect to an infinite tensor product state,

$$(\mathcal{M}, \psi) := (\mathcal{A} \otimes \mathcal{C}^{\otimes \mathbb{N}_0}, \varphi \otimes \chi^{\otimes \mathbb{N}_0})$$

is a probability space which can be equipped with a representation of the monoid of partial shifts S^+ and the Thompson monoid F^+ . For $n \in \mathbb{N}_0$, let β_n denote the partial shift which acts on the weak*-total set of finite elementary tensors in \mathcal{M} as

$$\beta_n(a \otimes x_0 \otimes \cdots \otimes x_{n-1} \otimes x_n \otimes x_{n+1} \otimes \cdots) := a \otimes x_0 \otimes \cdots \otimes x_{n-1} \otimes \mathbb{1}_{\mathcal{C}} \otimes x_n \otimes x_{n+1} \otimes \cdots.$$

Proposition 5.1.1. *The maps $h_n \mapsto \beta_n =: \varrho(h_n)$, with $n \in \mathbb{N}_0$, extend multiplicatively to a representation $\varrho: S^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ which has the generating property.*

Proof. Each β_n extends to a unital injective *-homomorphism on \mathcal{M} , denoted by the same symbol, such that $\psi \circ \beta_n = \psi$. As the modular automorphism group of (\mathcal{M}, ψ) equals the von Neumann algebraic tensor product of the modular automorphism groups of its tensor factors, i.e. $\sigma_t^\psi = \sigma_t^\varphi \otimes (\sigma_t^\chi)^{\otimes \mathbb{N}_0}$, it is easily verified that $\beta_n \sigma_t^\psi = \sigma_t^\psi \beta_n$ for all $t \in \mathbb{R}$. Thus $\beta_n \in \text{End}(\mathcal{M}, \psi)$ for all $n \in \mathbb{N}_0$. For $0 \leq k \leq \ell < \infty$, the relations $\beta_k \beta_\ell = \beta_{\ell+1} \beta_k$ are directly checked on elementary tensors. Consequently the endomorphisms β_0, β_1, \dots satisfy the relations of the monoid generators $h_0, h_1, \dots \in S^+$. Finally, the generating property of the representation ϱ is inferred from $\mathcal{A} \otimes \mathcal{C}^{\otimes n} \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0} \subset \mathcal{M}^{\beta_n}$ which ensures that the unital *-algebra $\bigcup_{n \in \mathbb{N}_0} \mathcal{M}^{\beta_n}$ is weak*-dense in \mathcal{M} . \square

Let $\epsilon: F^+ \rightarrow S^+$ be the monoid epimorphism with $\epsilon(g_n) = h_n$ for all $n \in \mathbb{N}$. Then

$$F^+ \ni g \mapsto \varrho \circ \epsilon(g) \in \text{End}(\mathcal{M}, \psi)$$

defines a representation of the Thompson monoid F^+ which also has the generating property. More general representations of F^+ can be constructed as follows.

Given the two random variables $C: (\mathcal{A}, \varphi) \rightarrow (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \chi)$ and $D: (\mathcal{C}, \chi) \rightarrow (\mathcal{C} \otimes \mathcal{C}, \chi \otimes \chi)$, let α_n denote the \mathbb{C} -linear extension of the map defined on a weak*-total subset of \mathcal{M} by

$$\alpha_n(a \otimes x_0 \otimes x_1 \otimes \cdots) := \begin{cases} C(a) \otimes x_0 \otimes x_1 \otimes \cdots & \text{if } n = 0, \\ a \otimes D(x_0) \otimes x_1 \otimes \cdots & \text{if } n = 1, \\ a \otimes x_0 \otimes \cdots \otimes D(x_{n-1}) \otimes \cdots & \text{if } n > 1. \end{cases} \quad (5.1.1)$$

Proposition 5.1.2. *The maps $g_n \mapsto \alpha_n =: \rho(g_n)$, with $n \in \mathbb{N}_0$, extend multiplicatively to a representation $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ which has the generating property.*

Proof. For $0 \leq k < \ell < \infty$, the relations $\alpha_k \alpha_\ell = \alpha_{\ell+1} \alpha_k$ are verified in a straightforward computation on finite elementary tensors of the form $x = a \otimes x_0 \otimes \cdots \otimes x_n \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}}$. We need to verify that each α_n commutes with the modular automorphism group $\sigma^\psi = \sigma^\varphi \otimes (\sigma^\chi)^{\otimes \mathbb{N}_0}$. As α_n is a *-homomorphism, this modular condition is ensured if

$$\alpha_n \circ \sigma_t^\psi(a \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0}) = \sigma_t^\psi \circ \alpha_n(a \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0}), \quad (5.1.2)$$

$$\alpha_n \circ \sigma_t^\psi(\mathbb{1}_{\mathcal{A}} \otimes y) = \sigma_t^\psi \circ \alpha_n(\mathbb{1}_{\mathcal{A}} \otimes y) \quad (5.1.3)$$

for all $a \in \mathcal{A}$ and $y \in \mathcal{C}^{\otimes \mathbb{N}_0}$. Let us first consider the case $n = 0$ in (5.1.2). Indeed, as the noncommutative random variable C intertwines the modular automorphism groups σ^φ and $\sigma^\varphi \otimes \sigma^\chi$,

$$\begin{aligned} \alpha_0 \circ \sigma_t^\psi(a \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0}) &= \alpha_0(\sigma_t^\varphi(a) \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0}) = (C \circ \sigma_t^\varphi(a)) \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}} \\ &= ((\sigma_t^\varphi \otimes \sigma_t^\chi) \circ C)(a) \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}} = \sigma_t^\psi(C(a) \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}}) \end{aligned}$$

$$= \sigma_t^\psi \circ \alpha_0(a \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0}).$$

Furthermore, again in the case $n = 0$, (5.1.3) is immediate from

$$\alpha_0 \circ \sigma_t^\psi(\mathbb{1}_{\mathcal{A}} \otimes y) = \beta_0 \circ \sigma_t^\psi(\mathbb{1}_{\mathcal{A}} \otimes y) = \sigma_t^\psi \circ \beta_0(\mathbb{1}_{\mathcal{A}} \otimes y),$$

since β_0 and σ_t^ψ commute by Proposition 5.1.1. Similar arguments ensure $\alpha_n \circ \sigma_t^\psi = \sigma_t^\psi \circ \alpha_n$ for $n \geq 1$. Finally, the generating property of ρ is inferred from $\mathcal{A} \otimes \mathcal{C}^{\otimes n-1} \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0} \subset \mathcal{M}^{\alpha_n}$ for all $n \geq 1$. \square

The representation ρ of F^+ may be considered as a perturbation of the representation ϱ of S^+ by locally acting operators C and D on the infinite tensor product factors of \mathcal{M} . To be more precise, the choice $D(x) = \mathbb{1}_{\mathcal{C}} \otimes x$ yields $\alpha_n = (\beta_{n-1}\beta_n^*)\beta_n = \beta_{n-1}$ for $n \geq 1$ and $\alpha_0 = (\alpha_0\beta_0^*)\beta_0$.

Theorem 5.1.3. *Let $\varrho: S^+ \mapsto \text{End}(\mathcal{M}, \psi)$ be the representation as introduced in Proposition 5.1.1. Then $(\mathcal{M}, \psi, \beta_0, \mathcal{M}^{\beta_1})$ is a (noncommutative) full Bernoulli shift with generator $\mathcal{M}^{\beta_1} = \mathcal{A} \otimes \mathcal{C} \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0}$.*

Proof. Let $\mathcal{B}_I := \bigvee_{i \in I} \beta_0^i(\mathcal{M}^{\beta_1})$ for $I \in \mathcal{I}(\mathbb{N}_0)$ and note that $\mathcal{B}_{[0,0]} = \mathcal{M}^{\beta_1}$. It is straightforward to check that $\mathcal{M}^{\beta_1} = \mathcal{A} \otimes \mathcal{C} \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0}$ and, more generally, $\mathcal{B}_{[m,n]} = \mathcal{A} \otimes \mathbb{1}_{\mathcal{C}}^{\otimes m} \otimes \mathcal{C}^{\otimes n-m+1} \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0}$ for $0 \leq m \leq n$. Since $\mathcal{B}_{\mathbb{N}_0} = \mathcal{M}$, the stationary process $(\mathcal{M}, \psi, \beta_0, \mathcal{M}^{\beta_1})$ is minimal. We are left to show that this minimal stationary process is actually a noncommutative Bernoulli shift (in the sense of Definition 2.6.15). Clearly, $\mathcal{M}^{\beta_0} \subset \mathcal{M}^{\beta_1}$ as $\mathcal{M}^{\beta_0} = \mathcal{A} \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0}$. We are left to verify the factorization

$$Q_0(xy) = Q_0(x)Q_0(y)$$

for any $x \in \mathcal{B}_I, y \in \mathcal{B}_J$ whenever $I \cap J = \emptyset$. Here Q_0 is the ψ -preserving normal conditional expectation from \mathcal{M} onto \mathcal{M}^{β_0} . As the conditional expectation Q_0 is of tensor type, i.e.

$$Q_0(a \otimes x_0 \otimes x_1 \otimes \cdots \otimes x_k \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0}) = a \otimes \chi(x_0)\chi(x_1) \cdots \chi(x_k)\mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0},$$

the required factorization easily follows. \square

Theorem 5.1.4. *Let $\rho: F^+ \mapsto \text{End}(\mathcal{M}, \psi)$ be a representation as introduced in Proposition 5.1.2. Then $(\mathcal{M}, \psi, \alpha_0, \mathcal{M}^{\alpha_1})$ is a stationary Markov process with generator \mathcal{M}^{α_1} . Moreover $(\mathcal{M}, \psi, \alpha_0, \mathcal{A}_0)$ is a stationary Markov process with generator $\mathcal{A}_0 := \mathcal{A} \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0} \subset \mathcal{M}^{\alpha_1}$.*

Proof. By Proposition 5.1.2, the representation ρ has the generating property. Now the Markovianity of the stationary process $(\mathcal{M}, \psi, \alpha_0, \mathcal{M}^{\alpha_1})$ follows from Corollary 4.3.7. We are left to show that the canonical local filtration of the stationary process $(\mathcal{M}, \psi, \alpha_0, \mathcal{A}_0)$ is Markovian. We note that the definition of the endomorphism α_0 is independent of the choice of the operator D in (5.1.1). Moreover, the inclusion $\mathcal{A}_0 := \mathcal{A} \otimes \mathbb{1}_{\mathcal{C}}^{\otimes \mathbb{N}_0} \subset \mathcal{M}^{\alpha_1}$ is valid for any choice of the operator D . Now Corollary 4.3.7 can again be applied to ensure Markovianity if there exists some $D: \mathbb{C} \rightarrow \mathbb{C} \otimes \mathbb{C}$ such that $\mathcal{A}_0 = \mathcal{M}^{\alpha_1}$. It is immediately verified that this equality occurs for the choice $D(x) = \mathbb{1}_{\mathcal{C}} \otimes x$. \square

These Markov processes may not be minimal. Note also that \mathcal{A}_0 may be strictly included in \mathcal{M}^{α_1} , as the latter depends on the choice of the operator D . For example, strict inclusion occurs for the choice $D(x) = x \otimes \mathbb{1}_{\mathcal{C}}$, but equality occurs for the choice $D(x) = \mathbb{1}_{\mathcal{C}} \otimes x$ in (5.1.1).

Remark 5.1.5. We remind the reader that the generating property of ρ and the relations of F^+ guarantee that the fixed point algebras \mathcal{M}^{α_n} form a tower of inclusions, even though we may not know explicitly what the fixed point algebras are. In particular, we get $\mathcal{M}^{\alpha_1} \subset \mathcal{M}^{\alpha_2}$. It is not obvious to see this directly (without using the relations of F^+) for a general operator D , as used in (5.1.1) for the definition of the endomorphisms α_n . However, the choice $D(x) = \mathbb{1}_{\mathcal{C}} \otimes x$ yields $\alpha_n = \beta_{n-1}$ for all $n \geq 1$. We infer from this that $\mathcal{M}^{\alpha_1} = \mathcal{M}^{\beta_0} = \mathcal{A} \otimes \mathbb{1}_{\mathcal{C}^{\otimes \mathbb{N}}} \subset \mathcal{A} \otimes \mathcal{C} \otimes \mathbb{1}_{\mathcal{C}^{\otimes \mathbb{N}}} = \mathcal{M}^{\beta_1} = \mathcal{M}^{\alpha_2}$. Similarly, choosing $D(x) = x \otimes \mathbb{1}_{\mathcal{C}}$, we get $\alpha_n = \beta_n$ for all $n \geq 0$; and the inclusion $\mathcal{M}^{\alpha_1} \subset \mathcal{M}^{\alpha_2}$ is clear from the inclusion of fixed point algebras of the partial shifts β_n . Finally, we note that if D is one of the above special random variables, then Markovianity can be proved without appealing to the Thompson monoid F^+ , see [39, Section 2.1].

Above, we directly constructed some representations of the Thompson monoid F^+ on infinite tensor products of noncommutative probability spaces and invoked some of our general results about such representations from Section 4 to obtain noncommutative stationary Markov processes. We present next a converse result which starts with a certain class of noncommutative stationary Markov processes for which we will make use of Proposition 2.6.10. Namely, we show that if a Markov map has a tensor dilation (in the terminology of Kümmerer [56], see Definition 2.6.8) then this Markov map can be obtained as the compression of a represented generator of the Thompson monoid F^+ .

Theorem 5.1.6. *Suppose $\gamma \in \text{End}(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \chi)$ and let ι_0 be the canonical embedding of (\mathcal{A}, φ) into $(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \chi)$. Then there exists a noncommutative probability space (\mathcal{M}, ψ) , a generating representation $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ and an embedding $\kappa: (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \chi) \rightarrow (\mathcal{M}, \psi)$ such that*

- (i) $\kappa(\mathcal{A} \otimes \mathbb{1}_{\mathcal{C}}) = \mathcal{M}^{\rho(g_1)}$,
- (ii) $\iota_0^* \gamma^n \iota_0 = \iota_0^* \kappa^* \rho(g_0^n) \kappa \iota_0$ for all $n \in \mathbb{N}_0$.

In particular, $(\mathcal{M}, \psi, \rho(g_0), \mathcal{M}^{\rho(g_1)})$ is a unilateral noncommutative stationary Markov process.

Proof. We take

$$(\mathcal{M}, \psi) := (\mathcal{A} \otimes \mathcal{C}^{\otimes \mathbb{N}_0}, \varphi \otimes \chi^{\otimes \mathbb{N}_0})$$

and construct a representation of the Thompson monoid F^+ as obtained in Proposition 5.1.2. That is, we define the representation $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ as $\rho(g_n) := \alpha_n$ as in (5.1.1) with $C: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{C}$ and $D: \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ given by $C(a) := \gamma(a \otimes \mathbb{1}_{\mathcal{C}})$ and $D(x) := \mathbb{1}_{\mathcal{C}} \otimes x$. Also recall that the partial shifts β_n can be constructed and it can be seen that $\gamma_0 \beta_0 = \alpha_0$, where γ_0 is the natural extension of γ to an endomorphism on (\mathcal{M}, ψ) . Let κ be the natural embedding of $(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \chi)$ into (\mathcal{M}, ψ) . The endomorphism γ_0 satisfies

$$\kappa^* \gamma_0^n \kappa \iota_0 = \gamma^n \iota_0 \quad \text{for all } n \in \mathbb{N}_0. \quad (5.1.4)$$

Note that for the case $n = 1$, the left hand side of this equation can be written as

$$\kappa^* \gamma_0 \kappa \iota_0 = \kappa^* \gamma_0 \beta_0 \kappa \iota_0 = \kappa^* \alpha_0 \kappa \iota_0. \quad (5.1.5)$$

Now, by Theorem 5.1.4, it follows that $(\mathcal{M}, \psi, \alpha_0, \mathcal{M}^{\alpha_1})$ is a noncommutative stationary Markov process, and $\kappa(\mathcal{A} \otimes \mathbb{1}_{\mathcal{C}}) = \mathcal{M}^{\alpha_1}$.

We note that $\kappa \iota_0 (\kappa \iota_0)^*$ is the ψ -preserving normal conditional expectation from \mathcal{M} onto $\mathcal{M}^{\alpha_1} = \kappa \iota_0 (\mathcal{A})$, and by definition, the stationary Markov process $(\mathcal{M}, \psi, \alpha_0, \mathcal{M}^{\alpha_1})$ has the transition operator

$$T := \kappa \iota_0 (\kappa \iota_0)^* \alpha_0 \kappa \iota_0 (\kappa \iota_0)^*.$$

We observe that (5.1.4) and (5.1.5) allow us to rewrite T as follows:

$$\begin{aligned} T &= \kappa \iota_0 (\kappa \iota_0)^* \alpha_0 \kappa \iota_0 (\kappa \iota_0)^* \\ &= \kappa \iota_0 \iota_0^* (\kappa^* \alpha_0 \kappa) (\kappa \iota_0)^* \\ &= \kappa \iota_0 \iota_0^* (\kappa^* \gamma_0 \kappa) \iota_0^* \kappa^* \\ &= \kappa \iota_0 \iota_0^* \gamma \iota_0 \iota_0^* \kappa^* \end{aligned} \quad (5.1.6)$$

On the other hand, Proposition 2.6.10 gives that T satisfies

$$T^n = \kappa \iota_0 (\kappa \iota_0)^* \alpha_0^n \kappa \iota_0 (\kappa \iota_0)^* \quad \text{for all } n \in \mathbb{N}_0. \quad (5.1.7)$$

Hence by (5.1.6) and (5.1.7),

$$\begin{aligned} (\kappa \iota_0 \iota_0^*) \gamma^n (\kappa \iota_0 \iota_0^*)^* &= [(\kappa \iota_0 \iota_0^*) \gamma (\kappa \iota_0 \iota_0^*)^*]^n \\ &= T^n = \kappa \iota_0 (\kappa \iota_0)^* \alpha_0^n \kappa \iota_0 (\kappa \iota_0)^*. \end{aligned}$$

Simplifying, we get

$$\iota_0^* \gamma^n \iota_0 = \iota_0^* \kappa^* \alpha_0^n \kappa \iota_0 \quad \text{for all } n \in \mathbb{N}_0,$$

as claimed in (ii) of the theorem. \square

Suppose $(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \chi, \alpha, \mathcal{A} \otimes \mathbb{1}_{\mathcal{C}})$ is a stationary Markov process. Then we just showed that α restricted to the generator $\mathcal{A} \otimes \mathbb{1}_{\mathcal{C}}$ can be obtained as the compression of a represented generator of F^+ . However, it is unknown in the generality of the present noncommutative setting if there exists a representation $\rho: F^+ \rightarrow \text{End}(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \chi)$ such that the Markov shift α equals the represented generator $\rho(g_0)$ itself. Thus it is unknown if the canonically associated stationary Markov sequence of random variables $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0}: (\mathcal{A}, \varphi) \rightarrow (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \chi)$ is maximal partially spreadable. We will see in the Subsection 5.2 that this canonically associated sequence ι is maximal partially spreadable in our algebraic framework for classical probability, see Theorem 5.2.4. Furthermore we will investigate in Subsection 5.3 an operator algebraic setting which allows to deduce that certain noncommutative stationary Markov sequences are partially spreadable.

5.2. Constructions in classical probability. The tensor product constructions from Subsection 5.1 apply of course to commutative von Neumann algebras (with separable predual) as they are of relevance in classical probability theory: a von Neumann algebra with separable predual is isomorphic to the essentially bounded functions on some standard probability space. Constructions and results on an algebraic reformulation of classical Markov processes were discussed in Section 3. Here we will provide the proof of the classical de Finetti theorem for stationary Markov sequences, Theorem 1.0.1, in its algebraic reformulation, Theorem 5.2.5.

We recall that a stationary Markov process is completely determined by its transition operator in classical probability, up to equivalence in distribution. This folklore result can be reformulated in the present operator algebraic setting as done next.

Proposition 5.2.1. Consider the two stationary Markov sequences $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ and $\tilde{\iota} \equiv (\tilde{\iota}_n)_{n \in \mathbb{N}_0} : (\tilde{\mathcal{A}}, \tilde{\varphi}) \rightarrow (\tilde{\mathcal{M}}, \tilde{\psi})$ with transition operators given by the φ -Markov maps on \mathcal{A} , R and \tilde{R} respectively. If \mathcal{M} and $\tilde{\mathcal{M}}$ are commutative von Neumann algebras, then the following are equivalent:

- (a) $\iota \stackrel{\text{distr}}{=} \tilde{\iota}$;
- (b) $R = \tilde{R}$.

Proof. (a) \implies (b): We conclude from Lemma 2.6.11 and from $\iota \stackrel{\text{distr}}{=} \tilde{\iota}$ that

$$\varphi(aR(b)) = \psi(\iota_0(a)\iota_1(b)) = \tilde{\psi}(\tilde{\iota}_0(a)\tilde{\iota}_1(b)) = \varphi(a\tilde{R}(b)).$$

for all $a, b \in \mathcal{A}$. But this implies $R = \tilde{R}$ by routine arguments.

(b) \implies (a): We need to show that $R = \tilde{R}$ implies

$$\psi(\iota_{k_1}(a_1) \cdots \iota_{k_n}(a_n)) = \tilde{\psi}(\tilde{\iota}_{k_1}(a_1) \cdots \tilde{\iota}_{k_n}(a_n))$$

for any $a_1, \dots, a_n \in \mathcal{A}$ and $k_1, \dots, k_n \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Since \mathcal{M} and $\tilde{\mathcal{M}}$ are commutative von Neumann algebras, and since random variables are (injective) $*$ -homomorphisms, we can assume $0 \leq k_1 < k_2 < \dots < k_n$ without loss of generality. We use Lemma 2.6.11 and $R = \tilde{R}$ to calculate

$$\begin{aligned} \psi(\iota_{k_1}(a_1) \cdots \iota_{k_n}(a_n)) &= \varphi(a_1 R^{k_2 - k_1}(a_2 \cdots R^{k_n - k_{n-1}}(a_n))) \\ &= \varphi(a_1 \tilde{R}^{k_2 - k_1}(a_2 \cdots \tilde{R}^{k_n - k_{n-1}}(a_n))) \\ &= \tilde{\psi}(\tilde{\iota}_{k_1}(a_1) \cdots \tilde{\iota}_{k_n}(a_n)). \end{aligned}$$

□

Notation 5.2.2. As in Subsection 3.4, throughout this subsection λ denotes the Lebesgue measure on the unit interval $[0, 1] \subset \mathbb{R}$ and the (non-)commutative probability space $(\mathcal{L}, \text{tr}_\lambda)$ is given by $\mathcal{L} := L^\infty([0, 1], \lambda)$ and $\text{tr}_\lambda := \int_{[0, 1]} \cdot d\lambda$.

We recall from Theorem 3.4.2 that given a probability space (\mathcal{A}, φ) , where \mathcal{A} is commutative and a φ -Markov map R on \mathcal{A} , there exists a Markov dilation $(\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_\lambda, \alpha, \iota_0)$ of R . Hence, as already seen before in the context of tensor product constructions, each Markov map in the present algebraic framework of classical probability can be obtained as the compression of a represented generator of the Thompson monoid F^+ .

Theorem 5.2.3. *Let (\mathcal{A}, φ) be a probability space where \mathcal{A} is commutative with separable predual, and let R be a φ -Markov map on \mathcal{A} . There exists a probability space (\mathcal{M}, ψ) , a generating representation $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ and an embedding $\iota: (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ such that*

- (i) $\iota(\mathcal{A}) = \mathcal{M}^{\rho(g_1)}$,
- (ii) $R^n = \iota^* \rho(g_0^n) \iota$ for all $n \in \mathbb{N}_0$.

Proof. By Theorem 3.4.2, there exists $\alpha \in \text{End}(\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_\lambda)$ such that $(\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_\lambda, \alpha, \mathcal{A} \otimes \mathbb{1}_{\mathcal{L}})$ is a stationary Markov process, and $R^n = \iota_0^* \alpha^n \iota_0$, for all $n \in \mathbb{N}_0$, where $\iota_0: (\mathcal{A}, \varphi) \rightarrow (\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_\lambda)$ denotes the canonical embedding $\iota_0(a) = a \otimes \mathbb{1}_{\mathcal{L}}$. The proof then follows from Theorem 5.1.6 by taking $\iota := \kappa \circ \iota_0$, as we get

$$R^n = \iota_0^* \alpha^n \iota_0 = \iota_0^* \kappa^* \rho(g_0)^n \kappa \iota_0 = \iota^* \rho(g_0^n) \iota \quad \text{for all } n \in \mathbb{N}_0.$$

□

Together with our general results from Section 4, our next result paves the road for a de Finetti theorem, Theorem 5.2.5, for (recurrent) stationary Markov sequences with values in a standard Borel space.

Theorem 5.2.4. *Let (\mathcal{A}, φ) and (\mathcal{M}, ψ) be probability spaces such that \mathcal{A} and \mathcal{M} are commutative. A stationary Markov sequence $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0}: (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ is (maximal) partially spreadable.*

Proof. We will show that there exists a probability space $(\tilde{\mathcal{M}}, \tilde{\psi})$ and a sequence $\tilde{\iota} \equiv (\tilde{\iota}_n)_{n \in \mathbb{N}_0}: (\mathcal{A}, \varphi) \rightarrow (\tilde{\mathcal{M}}, \tilde{\psi})$ which has the same distribution as the stationary Markov sequence ι and which satisfies, for all $n \in \mathbb{N}$,

$$\tilde{\rho}(g_n) \tilde{\iota}_0 = \tilde{\iota}_0 \quad \text{and} \quad \tilde{\rho}(g_0^n) \tilde{\iota}_0 = \tilde{\iota}_n$$

for some representation $\tilde{\rho}: F^+ \rightarrow \text{End}(\tilde{\mathcal{M}}, \tilde{\psi})$.

Since ι is stationary and Markovian there exist a φ -Markov map R on \mathcal{A} such that

$$\varphi(aR(b)) = \psi(\iota_0(a)\iota_1(b)).$$

By Theorem 5.2.3, there exists a generating representation $\tilde{\rho}: F^+ \rightarrow \text{End}(\tilde{\mathcal{M}}, \tilde{\psi})$, where $(\tilde{\mathcal{M}}, \tilde{\psi}) = (\mathcal{A} \otimes \mathcal{L}^{\otimes \mathbb{N}_0}, \varphi \otimes \text{tr}_\lambda^{\otimes \mathbb{N}_0})$, and an embedding $\kappa: (\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \text{tr}_\lambda) \rightarrow (\tilde{\mathcal{M}}, \tilde{\psi})$ such that $\kappa(\mathcal{A} \otimes \mathbb{1}_{\mathcal{L}}) = \tilde{\mathcal{M}}^{\tilde{\rho}(g_1)}$.

We infer from Theorem 5.1.4 that $(\tilde{\mathcal{M}}, \tilde{\psi}, \tilde{\rho}(g_0), \mathcal{A} \otimes \mathbb{1}_{\mathcal{L}^{\otimes \mathbb{N}_0}})$ is a stationary Markov process such that

$$\varphi(aR(b)) = \tilde{\psi}((a \otimes \mathbb{1}_{\mathcal{L}^{\otimes \mathbb{N}_0}}) \tilde{\rho}(g_0)(b \otimes \mathbb{1}_{\mathcal{L}^{\otimes \mathbb{N}_0}})).$$

Consequently, the sequence of random variables $\tilde{\iota} \equiv (\tilde{\iota}_n)_{n \in \mathbb{N}_0}: (\mathcal{A}, \varphi) \rightarrow (\tilde{\mathcal{M}}, \tilde{\psi})$, defined by

$$\tilde{\iota}_0(a) := a \otimes \mathbb{1}_{\mathcal{L}^{\otimes \mathbb{N}_0}} \quad \text{and} \quad \tilde{\iota}_n(a) := \tilde{\rho}(g_0)^n \iota_0(a) \quad \text{for } n > 0,$$

is Markovian and partially spreadable, both by construction. Additionally, $\tilde{\iota}$ is maximal partially spreadable as $\tilde{\iota}_0(\mathcal{A}) = \mathcal{A} \otimes \mathbb{1}_{\mathcal{L}^{\otimes \mathbb{N}_0}} = \tilde{\mathcal{M}}^{\tilde{\rho}(g_1)} = \tilde{\mathcal{M}}_0$. Furthermore, the sequences $\tilde{\iota}$ and ι have the same distribution, as they are stationary Markov sequences with the same Markov map R (see Proposition 5.2.1). □

Theorem 4.4.3 and Theorem 5.2.4 consolidate to the following result in the setting of commutative probability spaces.

Theorem 5.2.5. *Let (\mathcal{A}, φ) and (\mathcal{M}, ψ) be probability spaces such that \mathcal{A} and \mathcal{M} are commutative with separable predual. Let $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \rightarrow (\mathcal{M}, \psi)$ be a sequence of random variables. Then the following are equivalent:*

- (a) ι is a maximal partially spreadable sequence;
- (b) ι is a stationary Markov sequence.

Thus we have arrived at the algebraic formulation of a de Finetti theorem for stationary Markov sequences. Its traditional formulation in terms of classical random variables is available in Theorem 1.0.1 and discussed in Subsection 3.2.

5.3. Constructions in the framework of operator algebras. Kümmerer's approach to an operator algebraic theory of stationary Markov processes is based on the concept of a coupling representation (see [56, 58] for example). Here we adapt and refine this approach such that it provides a rich operator algebraic framework for the construction of representations of the Thompson monoid F^+ .

Our investigations are motivated by the elementary observation that the relations of the Thompson monoid F^+ are robust under certain algebraic 'perturbations' which we introduce and formalize next.

Definition 5.3.1. The *extended monoid* EF^+ is presented by the set of generators $\{g_n, c_n \mid n \in \mathbb{N}_0\}$ subject to the relations

$$g_k g_\ell = g_{\ell+1} g_k, \quad c_k c_{\ell+1} = c_{\ell+1} c_k, \quad c_k g_{\ell+1} = g_{\ell+1} c_k, \quad g_k c_\ell = c_{\ell+1} g_k$$

for every $0 \leq k < \ell < \infty$.

Evidently the first set of generators $\{g_n\}_{n \in \mathbb{N}_0}$ satisfies the relations of the Thompson monoid F^+ .

Proposition 5.3.2. *The submonoid $QF^+ := \langle c_n g_n \mid n \in \mathbb{N}_0 \rangle^+ \subset EF^+$ is a quotient of the monoid F^+ .*

Proof. An elementary computation, based on all defining relations of the monoid EF^+ , shows that the elements of the set $\{\tilde{g}_n := c_n g_n \mid n \in \mathbb{N}_0\}$ satisfy the relations of the Thompson monoid F^+ for $0 \leq k < \ell < \infty$:

$$\begin{aligned} \tilde{g}_k \tilde{g}_\ell &= c_k g_k c_\ell g_\ell = c_k c_{\ell+1} g_k g_\ell = c_{\ell+1} c_k g_k g_\ell \\ &= c_{\ell+1} c_k g_{\ell+1} g_k = c_{\ell+1} g_{\ell+1} c_k g_k = \tilde{g}_{\ell+1} \tilde{g}_k. \end{aligned}$$

□

Definition 5.3.3. The *extended monoid* ES^+ is presented by the set of generators $\{h_n, c_n \mid n \in \mathbb{N}_0\}$ subject to the relations

$$h_k h_\ell = h_{\ell+1} h_k, \quad c_k c_{\ell+1} = c_{\ell+1} c_k, \quad c_k h_{\ell+1} = h_{\ell+1} c_k, \quad h_k c_\ell = c_{\ell+1} h_k,$$

and $h_k h_k = h_{k+1} h_k$ for every $0 \leq k < \ell < \infty$.

Clearly the monoid ES^+ is a quotient of the monoid EF^+ , due to the additional set of relations for the h_k 's. The extended monoid ES^+ algebraically encodes certain local perturbations of the partial shifts monoid S^+ which, roughly phrasing, corresponds to the perturbation of Bernoulli shifts such that one obtains Markov shifts in classical probability.

Proposition 5.3.4. *The submonoid $QS^+ := \langle c_n h_n \mid n \in \mathbb{N}_0 \rangle^+ \subset ES^+$ is a quotient of the monoid F^+ .*

Proof. An elementary computation shows that the elements of the set $\{\tilde{g}_n := c_n h_n \mid n \in \mathbb{N}_0\}$ satisfy the relations of the Thompson monoid F^+ . \square

Remark 5.3.5. Actually the relations in Definition 5.3.1 have been identified by some reverse engineering: each c_k should provide a suitable ‘local perturbation’ of g_k such that $(c_k g_k)(c_\ell g_\ell) = (c_{\ell+1} g_{\ell+1})(c_k g_k)$ for $0 \leq k < \ell < \infty$. An alternative ‘perturbation’ is given by the extended monoid FF^+ which is defined to be presented by generators $\{c_n, g_n\}_{n \in \mathbb{N}_0}$ subject to the relations

$$g_k g_\ell = g_{\ell+1} g_k, \quad c_k c_\ell = c_{\ell+1} c_k, \quad c_k g_{\ell+1} = g_{\ell+1} c_k, \quad g_k c_\ell = c_\ell g_k$$

for every $(0 \leq k < \ell < \infty)$. Here both the c_k 's and the g_k 's satisfy the relations of the Thompson monoid F^+ and the last two sets of relations can be combined to a single set of relations on commutativity: $c_k g_\ell = g_\ell c_k$ whenever $k \notin \{\ell - 1, \ell\}$.

Remark 5.3.6. The results on semi-cosimplicial structures as obtained in [31] make it tempting to also investigate a more restrictive perturbed version for the partial shifts monoid S^+ . So let the *extended semi-cosimplicial monoid* ES_r^+ be presented by the set of generators $\{h_n, d_n \mid n \in \mathbb{N}_0\}$ subject to the relations

$$h_k h_\ell = h_{\ell+1} h_k, \quad d_k d_{\ell+1} = d_{\ell+1} d_k, \quad d_k h_{\ell+1} = h_{\ell+1} d_k, \quad h_k d_\ell = d_{\ell+1} h_k$$

for every $0 \leq k \leq \ell < \infty$. These relations ensure that the submonoid $QS_r^+ := \langle c_n g_n \mid n \in \mathbb{N}_0 \rangle \subset ES_r^+$ is a quotient of the monoid S^+ . In comparison to the extended monoid EF^+ , the additional relations are more restrictive for possible extensions of the monoid S^+ . Roughly phrasing, these additional relations encode algebraically the perturbative difference between Markovianity and stochastic independence in classical probability. We conjecture that QS_r^+ and S^+ are isomorphic as monoids.

Similarly as it was discovered for the monoid F^+ in Section 4, the representation theory of these extended monoids in the endomorphisms of a noncommutative probability space goes along with very rich structures of commuting squares. Here we restrict ourselves to present a single result, mainly in the intention to illustrate how Bernoulli shifts and, as their perturbation, Markov shifts can be simultaneously obtained from the representation theory of the extended monoid ES^+ .

Recall from Definition 2.6.2 that a noncommutative stationary process $(\mathcal{M}, \psi, \beta, \mathcal{A}_0)$ is said to be spreadable if the canonically associated sequence of random variables $(\iota_n)_{n \geq 0}: (\mathcal{A}_0, \psi_0) \rightarrow (\mathcal{M}, \psi)$ is spreadable, where $\iota_n := \beta^n|_{\mathcal{A}_0}$ and $\psi_0 = \psi|_{\mathcal{A}_0}$.

Theorem 5.3.7. *Suppose (\mathcal{M}, ψ) is equipped with a representation $\rho: ES^+ \rightarrow \text{End}(\mathcal{M}, \psi)$. Let $\mathcal{B}_0 := \mathcal{M}^{\rho(h_1)}$ and $(\mathcal{B}_\infty, \psi_\infty) := (\bigvee_{n \in \mathbb{N}_0} \rho(h_0^n)(\mathcal{B}_0), \psi|_{\mathcal{B}_\infty})$. Further let $\mathcal{A}_0 := \bigcap_{k \geq 1} \mathcal{M}^{\rho(c_k h_k)}$.*

- (i) The restricted represented generator $\beta := \rho(h_0)|_{\mathcal{B}_\infty}$ defines the spreadable Bernoulli shift $(\mathcal{B}_\infty, \psi_\infty, \beta, \mathcal{B}_0)$.
- (ii) The represented generator $\alpha := \rho(c_0 h_0)$ defines the (not necessarily minimal) stationary Markov process $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$.

If $\mathcal{M} = \mathcal{B}_\infty$ and $\mathcal{A}_0 = \mathcal{B}_\infty^\beta$, then the stationary Markov process $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$ has the coupling representation $(\mathcal{M}, \psi, \gamma\beta, \mathcal{A}_0)$, with coupling $\gamma := \rho(c_0)$.

Proof. (i) Let $\rho_B(h_n) := \beta_n := \rho(h_n), h_n \in S^+, n \in \mathbb{N}_0$. Then $\beta = \beta_0$ and ρ_B gives a representation of the monoid S^+ in $\text{End}(\mathcal{M}, \psi)$. Hence $(\mathcal{B}_\infty, \psi_\infty, \beta, \mathcal{B}_0)$ is spreadable (compare Definition 2.6.2). Also observe that $\mathcal{B}_\infty^\beta = \mathcal{B}_\infty^{\rho_B(h_0)} \subset \mathcal{B}_\infty^{\rho_B(h_1)} \subset \mathcal{B}_0$ due to the relations of S^+ . Now, the fact that it is a Bernoulli shift follows from [53, Theorem 8.2].

(ii) Let $\rho_M(g_n) := \alpha_n := \rho(c_n h_n), g_n \in F^+, n \in \mathbb{N}_0$. Then $\alpha = \alpha_0$ and ρ_M gives a representation of the monoid F^+ in $\text{End}(\mathcal{M}, \psi)$. Also observe that $\mathcal{M}_0^{\rho_M} := \bigcap_{k \geq 1} \mathcal{M}^{\rho_M(g_k)} = \bigcap_{k \geq 1} \mathcal{M}^{\rho(c_k h_k)} = \mathcal{A}_0$, hence by Corollary 4.3.5, $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$ is a stationary Markov process. \square

The significance of this result is that it indicates a promising strategy of how to construct a representation of the Thompson monoid F^+ from a large class of noncommutative stationary Markov processes. The starting point is the construction of a spreadable noncommutative Bernoulli shift which is known to be in a bijective correspondence to equivalence classes of spreadable sequences of noncommutative random variables (see [53, 31]). In other words, the construction of a spreadable Bernoulli shift amounts to the construction of a representation of the partial shift monoid S^+ . But as this monoid is a quotient of the Thompson monoid F^+ , spreadable Bernoulli shifts correspond to a particular class of representations of the Thompson monoid F^+ . Suitable perturbations of this particular class will provide certain Markov shifts and wider classes of representations of the Thompson monoid F^+ .

Proposition 5.3.8. *Suppose $(\mathcal{M}, \psi, \beta, \mathcal{B}_0)$ is a spreadable noncommutative Bernoulli shift. Then there exists a generating representation $\rho_\beta: S^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ such that $\beta = \rho_\beta(h_0)$ and $\mathcal{B}_0 \subset \mathcal{M}^{\rho_\beta(h_k)}$ for all $k \geq 1$.*

Proof. If $(\mathcal{M}, \psi, \beta, \mathcal{B}_0)$ is spreadable, then as it is a minimal noncommutative Bernoulli shift, there exists a representation $\rho_\beta: S^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ such that for $\lambda_n := \beta^n|_{\mathcal{B}_0}$ we get $\lambda_n = \rho_\beta(h_0^n)\lambda_0$ for all $n \in \mathbb{N}_0$ and $\rho_\beta(h_k)\lambda_0 = \lambda_0$ for all $k \geq 1$ (see [31, Theorem 4.5]). The representation ρ_β has the generating property by construction. \square

Definition 5.3.9. The representation $\rho_\beta: S^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ (as introduced in Proposition 5.3.8) is said to be *associated* to the spreadable Bernoulli shift $(\mathcal{M}, \psi, \beta, \mathcal{B}_0)$.

Corollary 5.3.10. A spreadable Bernoulli shift $(\mathcal{M}, \psi, \beta, \mathcal{B}_0)$ is partially spreadable.

Proof. As in Proposition 5.3.8, let ρ_β be the representation associated to the spreadable noncommutative Bernoulli shift. Denote by $\epsilon: S^+ \rightarrow F^+$ the canonical epimorphism which maps the generator $g_k \in F^+$ to the generator $h_k \in S^+$ for all $k \in \mathbb{N}_0$. Then $\rho := \rho_\beta \circ \epsilon$ defines a representation of F^+ such that the canonically associated sequence

of random variables $(\lambda_n)_{n \geq 0}$ (as used in the proof of Proposition 5.3.8) is partially spreadable. \square

A large class of noncommutative Markov shifts can be obtained as certain perturbations of noncommutative Bernoulli shifts, as developed and investigated by Kümmerer in [56, 57]. We refine the notion of a coupling representation so that it applies to spreadable noncommutative Bernoulli shifts.

Definition 5.3.11. A sequence $(\gamma_n)_{n \geq 0} \in \text{End}(\mathcal{M}, \psi)$ is called a *coupling (sequence)* to a spreadable Bernoulli shift $(\mathcal{M}, \psi, \beta, \mathcal{B}_0)$ with associated representation $\rho: S^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ if, for all $0 \leq k < \ell < \infty$,

$$\rho(h_k) \gamma_\ell = \gamma_{\ell+1} \rho(h_k), \quad \gamma_k \rho(h_{\ell+1}) = \rho(h_{\ell+1}) \gamma_k, \quad \gamma_k \gamma_{\ell+1} = \gamma_{\ell+1} \gamma_k.$$

Proposition 5.3.12. Let $(\gamma_n)_{n \geq 0}$ be a coupling sequence to the spreadable Bernoulli shift $(\mathcal{M}, \psi, \beta, \mathcal{B}_0)$ with associated representation $\rho_\beta: S^+ \rightarrow \text{End}(\mathcal{M}, \psi)$. Then a representation $\rho: F^+ \rightarrow \text{End}(\mathcal{M}, \psi)$ is defined by the multiplicative extension of

$$F^+ \ni g_k \mapsto \gamma_k \rho_\beta \epsilon(g_k) \in \text{End}(\mathcal{M}, \psi) \quad (0 \leq k < \infty).$$

Here ϵ denotes the canonical epimorphism from F^+ onto S^+ .

Proof. The relations of the Thompson monoid F^+ are satisfied by $\gamma_k \rho_\beta \epsilon(g_k)$ as for $0 \leq k < \ell < \infty$, the definition of a coupling sequence ensures that

$$\begin{aligned} (\gamma_k \rho_\beta \epsilon(g_k)) (\gamma_\ell \rho_\beta \epsilon(g_\ell)) &= (\gamma_k \rho_\beta(h_k)) (\gamma_\ell \rho_\beta(h_\ell)) \\ &= \gamma_k \gamma_{\ell+1} \rho_\beta(h_k) \rho_\beta(h_\ell) \\ &= \gamma_{\ell+1} \gamma_k \rho_\beta(h_{\ell+1}) \rho_\beta(h_k) \\ &= \gamma_{\ell+1} \rho_\beta(h_{\ell+1}) \gamma_k \rho_\beta(h_k) \\ &= (\gamma_{\ell+1} \rho_\beta \epsilon(g_{\ell+1})) (\gamma_k \rho_\beta \epsilon(g_k)). \end{aligned}$$

\square

Remark 5.3.13. It is known that there exist non-spreadable noncommutative Bernoulli shifts (see [53]). We conjecture that there exist also noncommutative Bernoulli shifts without partial spreadability. An affirmative answer to this conjecture implies that there exist noncommutative Markov shifts beyond the representation theory of the Thompson monoid F^+ .

ACKNOWLEDGEMENTS

The first author would like to thank the Isaac Newton Institute for Mathematical Sciences in Cambridge, England, for support and hospitality during the programme *Operator Algebras: Subfactors and their Applications* in the spring of 2017 where some initial studies for this paper were undertaken. The second author was supported by a Government of Ireland Postdoctoral Fellowship (Project ID: GOIPD/2018/498). The first and second author acknowledge several helpful discussions with B. V. Rajarama Bhat in an early stage of this project. The first and third author would like to thank Persi Diaconis for stimulating discussions on Markov chains during his visit at University College Cork in September 2018. The first author would like to thank Gwion

Evans, Rolf Gohm, Burkhard Kümmerer and Hans Maassen for fruitful discussions on noncommutative Markov processes.

REFERENCES

- [1] Accardi, L.; Frigerio, A.; and Lewis, J. T. (1982). Quantum stochastic processes. *Publ. Res. Inst. Math. Sci.* 18 (1), 97–133.
- [2] Accardi, L.; Ghorbal, A. B.; Crismale, V.; and Lu, Y. G. (2008). Singleton conditions and quantum de Finetti's theorem. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 11 (4), 639–660.
- [3] Aiello, V.; and Jones, V. F. R. (2021). On spectral measures for certain unitary representations of R. Thompson's group F . *J. Funct. Anal.* 280 (1), 108777, 1–27.
- [4] Aldous, D. J. (1985). Exchangeability and related topics. *École d'Été de Probabilités de Saint-Flour XIII – 1983. Lect. Notes in Math.* Springer, Berlin.
- [5] Anantharaman-Delaroche, C. (2006). On ergodic theorems for free group actions on noncommutative spaces. *Probab. Theory Relat. Fields* 135 (4), 115–130.
- [6] Attal, S. (2010) Markov chains and dynamical systems: the open system point of view. *Commun. Stoch. Anal.* 4 (4), 523–540.
- [7] Banica, T.; Curran, S.; and Speicher, R. (2012). De Finetti theorems for easy quantum groups. *Ann. Probab.* 40 (1), 401–435.
- [8] Belk, J. (2004). Thompson's group F . *PhD Thesis*, Cornell University.
- [9] Brothier, A.; and Jones, V. F. R. (2019). Pythagorean representations of Thompson's groups. *J. Funct. Anal.* 277 (7), 2442–2469.
- [10] Brothier, A.; and Jones, V. F. R. (2019). On the Haagerup and Kazhdan properties of R. Thompson's groups. *J. Group Theory* 22 (5), 795–807.
- [11] Bu, K.; Jaffe, A.; Liu, Z.; and Wu, J. (2020). De Finetti theorems for braided parafermions. *Comm. Math. Phys.* 373 (1), 435–456.
- [12] Burillo, J. (2004). Growth of positive words in Thompson's group F . *Communications in Algebra* 32 (8), 3087–3094.
- [13] Burke, C. J.; and Rosenblatt, M. (1958). A Markovian function of a Markov chain. *Ann. of Math. Stat.* 29 (4), 1112–1122.
- [14] Crismale, V.; and Fidaleo, F. (2012). De Finetti theorem on the CAR algebra. *Comm. Math. Phys.* 315 (1), 135–152.
- [15] Crismale, V.; and Fidaleo, F. (2015). Exchangeable stochastic processes and symmetric states in quantum probability. *Ann. Mat. Pura Appl.* 194 (4), 969–993.
- [16] Crismale, V.; Fidaleo, F.; and Griseta, M. E. (2020). Spreadability for quantum stochastic processes, with an application to boolean commutation relations. *Entropy* 22 (5), 532.
- [17] Crismale, V.; Rossi, S.; and Zurlo, P. (2022). De Finetti-type theorems on quasi-local algebras and infinite Fermi tensor products. *eprint arXiv:2201.02488*.
- [18] Curran, S. (2009). Quantum exchangeable sequences of algebras. *Indiana Univ. Math. J.* 58 (3), 1097–1125.
- [19] Curran, S. (2010). Quantum rotatability. *Trans. Amer. Math. Soc.* 362 (9), 4831–4851.
- [20] Curran, S. (2011). A characterization of freeness by invariance under quantum spreading. *J. Reine Angew. Math.* 659, 43–65.
- [21] Dabrowski, Y.; Dykema, K. J.; and Mukherjee, K. (2014). The simplex of tracial quantum symmetric states. *Studia Math.* 225 (3), 203–218.
- [22] Dehornoy, P.; and Tesson, E. (2019). Garside combinatorics for Thompson's monoid F^+ and a hybrid with the braid monoid B_∞^+ . *Algebraic Combinatorics* 2 (4), 683–709.
- [23] Dharmadhikari, S. W. (1963). Functions of finite Markov chains. *Ann. Math. Stat.* 34 (3), 1022–1032.
- [24] Dharmadhikari, S. W. (1963). Sufficient conditions for a stationary process to be a function of a finite Markov chain. *Ann. Math. Stat.* 34 (3), 1033–1041.

- [25] Dharmadhikari, S. W. (1964). Exchangeable processes which are functions of stationary Markov chains. *Ann. Math. Stat.* 35 (1), 429–430.
- [26] Diaconis, P.; and Freedman, D. (1980). De Finetti’s theorem for Markov chains. *Ann. Probab.* 8 (1), 115–130.
- [27] Diaconis, P. (1988). Recent progress on de Finetti’s notions of exchangeability. *Bayesian statistics* 3, 111–125.
- [28] Dykema, K. J.; and Köstler, C. (2014). Tail algebras of quantum exchangeable random variables. *Proc. Amer. Math. Soc.* 142 (1), 3853–3863.
- [29] Dykema, K. J.; Köstler, C.; and Williams, J. D. (2017). Quantum symmetric states on free product C^* -algebras. *Trans. Amer. Math. Soc.* 369, 645–679.
- [30] Dykema, K.; and Mukherjee, K. (2017). KMS quantum symmetric states. *J. Math. Phys.* 58 (1), 012103.
- [31] Evans, D. G.; Gohm, R.; and Köstler, C. (2017). Semi-cosimplicial objects and spreadability. *Rocky Mountain J. Math.* 47 (6), 1839–1873.
- [32] Fidaleo, F. (2022). Symmetric states for C^* -Fermi systems I: De Finetti theorem. *eprint arXiv:2201.00421*.
- [33] Fidaleo, F. (2022). Symmetric states for C^* -Fermi systems II: Klein transformation and their structure. *eprint arXiv:2202.03709*.
- [34] Finesso, L.; and Prosdocimi, C. (2009). Partially exchangeable hidden Markov models. *2009 European Control Conference (ECC)* 3910–3914.
- [35] Finesso, L.; and Prosdocimi, C. (2016). Countable partially exchangeable mixtures. *J. Theoret. Probab.* 29, no. 2, 423–442.
- [36] Freedman, D. A. (1962). Invariants under mixing which generalize de Finetti’s theorem. *Ann. Math. Statist.* 33 (3), 916–923.
- [37] Freslon, A.; and Weber, M. (2016). On bi-free de Finetti theorems. *Ann. Math. Blaise Pascal* 23 (1), 21–51.
- [38] Gärtner, A.; and Kümmerer, B. (2012). A coherent approach to recurrence and transience for quantum Markov operators. *eprint arXiv:1211.6876*.
- [39] Gohm, R. (2004). Noncommutative stationary processes. Springer.
- [40] Gohm, R.; and Köstler, C. (2009). Noncommutative independence from the braid group \mathbb{B}_∞ . *Comm. Math. Phys.* 289, 435–482.
- [41] Gohm, R.; and Köstler, C. (2010). Noncommutative independence from characters of the infinite symmetric group \mathbb{S}_∞ . *eprint arXiv:1005.5726v1*.
- [42] Gohm, R.; and Köstler, C. (2012). Noncommutative independence in the infinite braid and symmetric group. *Noncommutative harmonic analysis with applications to probability III, Banach Center Publ., Polish Acad. Sci. Inst. Math., Warsaw* 96, 193–206.
- [43] Goodman, F. M.; de la Harpe, P.; and Jones, V. F. R. (1989). Coxeter Graphs and Towers of Algebras. Springer-Verlag.
- [44] Haagerup, U.; and Musat, M. (2011). Factorization and dilation problems for completely positive maps on von Neumann algebras. *Comm. Math. Phys.* 303 (2), 555–594.
- [45] Hewitt, E.; and Savage, L. J. (1955). Symmetric measures on Cartesian products. *Trans. Amer. Math. Soc.* 80, 470–501.
- [46] Jones, V. F. R. (2017). Some unitary representations of Thompson’s groups F and T . *J. Comb. Algebra* 1 (1), 1–44.
- [47] Jones, V. F. R. (2018). A no-go theorem for the continuum limit of a periodic quantum spin chain. *Comm. Math. Phys.* 357 (1), 295–317.
- [48] Jones, V. F. R. (2018). Scale invariant transfer matrices and Hamiltonians. *J. Phys. A* 51 (10), 104001.
- [49] Jones, V. F. R.; and Sunder, V. S. (1997). Introduction to Subfactors. London Mathematical Society Lecture Note Series (234). Cambridge University Press.
- [50] Kallenberg, O. (1982). Characterizations and embedding properties in exchangeability. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 60, 249–281.

- [51] Kallenberg, O. (1997). Foundations of Modern Probability. Springer.
- [52] Kallenberg, O. (2005). Probabilistic Symmetries and Invariance Principles. Springer-Verlag.
- [53] Köstler, C. (2010). A noncommutative extended de Finetti theorem. *J. Funct. Anal.* 258 (4), 1073–1120.
- [54] Köstler, C.; Krishnan, A. (2022). Markovianity and the Thompson group F . *SIGMA* 18, 083, 27 pages, *eprint* arXiv:2204.03595.
- [55] Köstler, C.; and Speicher, R. (2009). A noncommutative de Finetti theorem: Invariance under quantum permutations is equivalent to freeness with amalgamation. *Comm. Math. Phys.*, 291, 473–490.
- [56] Kümmerer, B. (1985). Markov dilations on W^* -algebras. *J. Funct. Anal.* 63, 139–177.
- [57] Kümmerer, B. (1986). Construction and structure of Markov dilations on W^* -algebras. *Habilitationsschrift*, Tübingen.
- [58] Kümmerer, B. (1993). Stochastic Processes with values in M_n as couplings to free evolutions. *Preprint*, Tübingen.
- [59] Kümmerer, B. (2021). Personal communication.
- [60] Kümmerer, B.; and Schröder, W. (1983). A Markov dilation of a non-quasifree Bloch evolution. *Comm. Math. Phys.* 90, 251–262.
- [61] Liu, W. (2015). A noncommutative de Finetti theorem for boolean independence. *J. Funct. Anal.* 269 (7), 1950–1994.
- [62] Liu, W. (2018). Extended de Finetti theorems for boolean independence and monotone independence. *Trans. Amer. Math. Soc.* 370 (3), 1959–2003.
- [63] Liu, W. (2019). General de Finetti type theorems in noncommutative probability. *Comm. Math. Phys.* 369 (3), 837–866.
- [64] Popa, S. (1989). Relative dimension, towers of projections and commuting squares of subfactors. *Pac. J. Math.* 137 (1), 181–207.
- [65] Rosenblatt, M. (1974). Random Processes. Springer.
- [66] Royden, H. L. (1968). Real Analysis. Collier Mac Millan.
- [67] Ryll-Nardzewski, C. (1957). On stationary sequences of random variables and the de Finetti's equivalence. *Colloq. Math.*, 4, 149–156.
- [68] Størmer, E. (1969). Symmetric states of infinite tensor products of C^* -algebras. *J. Funct. Anal.* 3 (1), 48–68.
- [69] Takesaki, M. (1972). Conditional expectations in von Neumann algebras. *J. Funct. Anal.* 9, 306–321.
- [70] Takesaki, M. (1979). Theory of Operator Algebras I. Springer.
- [71] Takesaki, M. (2003). Theory of Operator Algebras II. Encyclopaedia Math. Sci., Springer.
- [72] Varilly, J. C. (1981). Dilation of a non-quasifree dissipative evolution. *Lett. Math. Phys.* 5, 113–116.
- [73] Vershik, A. M. (2017). The theory of filtrations of subalgebras, standardness, and independence. *Russian Math. Surveys* 72 (2), 257–333.

C. KÖSTLER, SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY COLLEGE CORK, CORK, IRELAND

Email address: `claus@ucc.ie`

A. KRISHNAN, DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, ONTARIO, CANADA

Email address: `arundhathi.krishnan@uwaterloo.ca`

S. J. WILLS, SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY COLLEGE CORK, CORK, IRELAND

Email address: `s.wills@ucc.ie`