Mary Immaculate College

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Postgraduate Dissertation

Classification of Affine and Projective Conics

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Abstract

Algebraic geometry is an area of Mathematics that looks at solving geometric problems using algebraic methods. We can do this by looking at the zeros of polynomials in different spaces. The aim of this dissertation is to construct morphisms into affine and projective spaces and classify conics in $\mathbf{A}^2$ and $\mathbf{P}^2$.

Introduction

Algebraic geometry is an area of Mathematics that looks at solving geometric problems using algebraic methods. We can do this by looking at the zeros of polynomials. The following Masters will introduce the concept of algebraic geometry. In order to achieve this we look first at the basic properties in topology, studying affine varieties, projective varieties and finally at how we construct morphisms with a goal of constructing morphisms into projective space. This will lead to classifying conics in $\mathbf{A}^2$ and $\mathbf{P}^2$. It is assumed that the reader has a basic knowledge and understanding of set theory and abstract algebra.

For this dissertation the theory I used and the structure I followed was predominantly from Robin Hartshornes book 'Algebraic Geometry'. I fill in the missing proofs and complete some exercises in the hope of giving the reader a more in depth understanding of the basics of Algebraic Geometry.

The first area of study is topology. Section 1 will introduce the notion of topology and topological spaces which we will be working in.

Section 2 will introduce the Affine n-space, define zero sets of polynomials, the relationship between ideals of the polynomial ring and subsets of affine n-space and finally it will define affine varieties.

Section 3 studies the Projective n- space. It will look at what defines a function on projective space, homogeneous polynomials and their zero sets before ultimately defining projective varieties.

Section 4 introduces the concept of localization of rings.

Section 5 discusses the mappings that are allowed between varieties and introduces regular functions.

Section 6 is where the main goal of this dissertation is discussed. This is the section where we study how to construct morphisms into projective space and classify conics in $\mathbf{A}^2$ and $\mathbf{P}^2$.

Contents

1 Topology and mappings

1.1 Topological space

Definition 1.1. A *topology* on a set X is a non-empty collection of subsets of X, called open sets, such that any union of open sets is open, any finite intersection of open sets is open, and both X and the empty set are open.

A set together with a topology is called a *topological space*.

Before we discuss topology further it is useful to recall the following properties in set theory known as De Morgan's Laws.

1 $(A \cup B)^c = A^c \cap B^c$ where $(A \cup B)^c$ is the complement of A union B.

2 $(A \cap B)^c = A^c \cup B^c$ where $(A \cap B)^c$ is the complement of A intersection B.

3 $(\bigcup_{A \in T} A)^c = \bigcap_{A \in T} A^c$.

4 $(\bigcap_{A \in T} A)^c = \bigcup_{A \in T} A^c$.

with $A \subset X$ and T is a set of subsets of X, where X is a set.

Remark. Let X be a set and $A \subseteq X$. Note here that A^c means the complement of A. i.e everything not in A. It can also be expressed as $X \backslash A$ or $X - A$. These are used interchangeably throughout the text so I make note of it here.

1.1.1 Standard topology on $\mathbb{R}^n$

Let $U \subset \mathbb{R}$ be a non-empty open subset. Here $\mathbb{R}$ is the real number line. Given $x \in U$, suppose there exists an open interval (a, b) such that $x \in (a, b)$ and $(a, b) \subset U$. Can we find ϵ such that every number y with $x - \epsilon < y < x + \epsilon$ lies in (a, b)? We can if we take the lesser value of $x - a$ and $b - x$.
Suppose $x - a < b - x$. Then we have $x - (x - a) < y < x + (x - a)$ which gives $a < y < x + x - a$ but we know $x - a < b - x$. Hence $x - a + x < b$. So we have $a < y < b$. Similarly if $b - x < x - a$ we have $x - (b - x) < y < x + (b - x)$. This gives $x - b + x < y < b$. But we know $b - x < x - a$ and therefore $a < x + x - b$. Hence $a < y < b$. So we can always find ϵ just by taking the lesser value of $x - a$ and $b - x$.
Note: In the case of $x - a = b - x$, we simply substitute in say $x - a$ and we get $x - (x - a) < y < x + (x - a)$ which gives $a < y < x + x - a$ but we know $x - a = b - x$ hence $x - a + x = b$. So we have $a < y < b$.

The *standard topology* on $\mathbb{R}$, the real number line, is characterized as follows.

A set U is open, if given $x \in U$, we can always find a positive real number ϵ such that every number y with $x - \epsilon < y < x + \epsilon$ lies in U. In other words a non-empty subset $U \subset \mathbb{R}$ is open if for each $x \in U$, there exists an open interval (a, b) such that $x \in (a, b)$ and $(a, b) \subset U$.

Lemma 1.2. *The standard topology on $\mathbb{R}$ is a topology.*

Proof. Let I be an index set. For every $i \in I$ let $U_i \subset \mathbb{R}$ be open. First we show the union of such open U_i is open in $\mathbb{R}$. Let $x \in \bigcup U_i$. Then x is contained in at least one of these U_i. But each of these U_i are open in $\mathbb{R}$. Hence there exists an open interval (a, b) such that $x \in (a, b)$ and $(a, b) \subset U_i$. But $U_i \subset \bigcup U_i$. Hence $(a, b) \subset \bigcup U_i$. Therefore $\bigcup U_i$ is open in $\mathbb{R}$.
Let U, V be open subsets of $\mathbb{R}$. Let $x \in U \cap V$. Then $x \in U$ and $x \in V$. Given that $x \in U$, there exists an open interval (a, b) with $x \in (a, b)$ and $(a, b) \subset U$. Given that $x \in V$ there exists an open interval (c, d) with $x \in (c, d)$ and $(c, d) \subset V$.
Put $p = \max(a, c)$ and $q = \min\ (b, d)$. (p, q) is an open interval. We will show $x \in (p, q)$ and $(p, q) \subset U \cap V$.
Given that $x \in (a, b)$ we have that $a < x < b$. Given that $x \in (c, d)$ we have that $c < x < d$. Since x is greater than both a and c we have that $x > p$. Given that x is less than b and d we have that $x < q$. Therefore $x \in (p, q)$.
Let $y \in (p, q)$. Then $p < y < q$. Therefore $a < y$, $c < y$, $y < b$ and $y < d$. Therefore $y \in (a, b) \subset U$ and $y \in (c, d) \subset V$. Hence $y \in U \cap V$. Therefore $(p, q) \subset U \cap V$.

Therefore $U \cap V$ is open in $\mathbb{R}$. This can be done for any finite intersection of open sets of $\mathbb{R}$.
By definition of the standard topology on $\mathbb{R}$ a subset of $\mathbb{R}$ is open if for each $x \in U$, there exists an open interval (a, b) such that $x \in (a, b)$ and $(a, b) \subset U$. Let $U = \emptyset$. There are no elements in $\emptyset$ hence $\emptyset$ satisfies the condition and is therefore open in $\mathbb{R}$.
By definition of the standard topology on $\mathbb{R}$, $\mathbb{R} \subset \mathbb{R}$ is open if for each $x \in \mathbb{R}$, there exists an open interval (a, b) such that $x \in (a, b)$ and $(a, b) \subset \mathbb{R}$. It is trivially true that $\mathbb{R}$ is open in $\mathbb{R}$, since for any $x \in \mathbb{R}$ we have the open interval $(-\infty, \infty) \subset \mathbb{R}$ which contains all x.
Therefore the standard topology on $\mathbb{R}$ is a topology. □

The standard topology on $\mathbb{R}^2$, the Euclidean plane, is characterized as a set U is open if given $u \in U$ we can always find a positive real number ϵ such that all points (x, y) in the disc with centre u and radius ϵ lie entirely in U. i.e All points in the disc whose distance from u is less than ϵ.

Lemma 1.3. *The standard topology on $\mathbb{R}^2$ is a topology.*

Proof. Let I be an index set. For every $i \in I$ let $U_i \subset \mathbb{R}^2$ be open. Let $u \in \bigcup U_i$. Then u is in at least one open U_i. Hence there is a positive ϵ such that all points (x, y) in the disc with centre u and radius ϵ lie entirely in U_i. But $U_i \subset \bigcup U_i$. Hence for $u \in \bigcup U_i$ we have found there exists positive ϵ such that all points (x, y) in the disc with centre u and radius ϵ lie entirely in $\bigcup U_i$.

Let U, V be open subsets of $\mathbb{R}^2$. Let $a \in U \cap V$. Then $a \in U$ and $a \in V$. Given that $a \in U$, then there exists positive ϵ such that all points (x, y) in the disc with centre u and radius ϵ lie entirely in U. Given that $a \in V$, then there exists positive ϵ' such that all points (x', y') in the disc with centre u and radius ϵ' lie entirely in V.

If $\epsilon = \epsilon'$ then the disc with centre u and radius ϵ lies entirely in $U \cap V$. If $\epsilon > \epsilon'$ then the disc with centre u and radius $\epsilon - \epsilon'$ lies entirely in $U \cap V$. If $\epsilon' > \epsilon$ then the disc with centre u and radius $\epsilon' - \epsilon$ lies entirely in $U \cap V$. Hence $U \cap V$ is open in $\mathbb{R}^2$.

By definition of the standard topology on $\mathbb{R}^2$, $\emptyset$ is open as there are no elements in $\emptyset$. Hence $\emptyset$ satisfies the condition.
By definition of the standard topology on $\mathbb{R}^2$, $\mathbb{R}^2 \subset \mathbb{R}^2$ is open if for each $a \in \mathbb{R}^2$, we can find a positive ϵ such that all points (x, y) in the disc with centre a and radius ϵ lie in $\mathbb{R}^2$. We let $\epsilon = 1$. Hence $\mathbb{R}^2$ is open.

Therefore the standard topology on $\mathbb{R}^2$ is a topology. □

The proof for the standard topology on $\mathbb{R}^n$ is similar to the proof of the standard topology on $\mathbb{R}^2$. We simply replace disc with ball.

Definition 1.4. In general we define the *standard topology on* $\mathbb{R}^n$ as follows. A set U is open, if given $u \in U$, we can always find a positive real number ϵ such that all points in the ball with centre u and radius ϵ lies entirely in U.

Let U be an open subset of $\mathbb{R}^2$ with the standard topology. Then for every $x \in U$ there is an open disc denoted D_x around x which is contained in U.

Lemma 1.5. *The union of all D_x over $x \in U$ is equal to U.*

Proof. $\bigcup_{x \in U} D_x \subseteq U$. This is given. So we just need to show $U \subseteq \bigcup_{x \in U} D_x$. Let $y \in U$. Hence there is a disc D_y around y which is contained in U. Hence y is also contained in the union. Therefore $U \subseteq \bigcup_{x \in U} D_x$. Hence $\bigcup_{x \in U} D_x = U$. □

Likewise for the standard topology of $\mathbb{R}$, a subset is open only if and only if it is the union of open intervals.

Definition 1.6. A subset $S \subseteq X$ is closed if its complement $X \backslash S$ is open.

Example 1.7. The set of integers $\mathbb{Z}$ is closed in $\mathbb{R}$. Let $(n, n+1), n \in \mathbb{Z}$ be an open interval in $\mathbb{R} \backslash \mathbb{Z}$. For any $r \in \mathbb{R}$ we can find $n \in \mathbb{Z}$ such that $n \leq r \leq n+1$. Let $r \in \mathbb{R} \backslash \mathbb{Z}$. Then $n < r < n+1$. r is in the open interval from n to $n+1$. Hence $\mathbb{R} \backslash \mathbb{Z}$ is just the union of such open intervals and hence an open set. Therefore $\mathbb{Z}$ is closed in $\mathbb{R}$.

Lemma 1.8. *Let X be a topological space. Let U, V be closed sets of X. $U \cup V$ is closed in X.*

Proof. Let U, V be closed subsets of X. Then by definition U^c and V^c are both open in X. Since X is a topological space $U^c \cap V^c$ is open in X. $U^c \cap V^c = (U \cup V)^c$ by De Morgan's law 1. Hence $(U \cup V)^c$ is open in X. Therefore $((U \cup V)^c)^c$ is closed in X. But $((U \cup V)^c)^c = U \cup V$. Hence $U \cup V$ is closed in X. □

Definition 1.9. A topological space X is called *Noetherian* if it satisfies the descending chain condition for closed subsets. That is for any sequence $Y_1 \supseteq Y_2 \supseteq \dots$ with Y_i a closed subset of X, there is an integer r such that $Y_r = Y_{r+1} = \dots$.

Theorem 1.10. *Let X be a topological space. The intersection of closed subsets of X is itself closed.*

Proof. Let T be a set of closed subsets of X. Let D be the intersection of this set of closed subsets.

$$D = \bigcap_{A \in T} A$$

The complement of D is

$$D^c = (\bigcap_{A \in T} A)^c$$

De Morgans Law 4 gives us

$$(\bigcap_{A \in T} A)^c = \bigcup_{A \in T} A^c$$

We know for all $A \in T$, A is closed. Hence A^c is open. X is a topological space. Hence by definition the union of open subsets is open. Therefore D^c is open. This means the complement of D^c is closed. But $(D^c)^c = D$. Recall D was defined as the intersection of closed subsets of X. Hence the intersection of closed subsets of a topological space X is closed. □

Definition 1.11. The closure of a subset A of X is defined as the intersection of all closed subsets which contain A. We denote the closure of A as $\bar{A}$. Theorem 1.10 tells us this closure is itself closed.

Definition 1.12. A subset S of X is dense in X if the closure of S is all of X.

Lemma 1.13. *Let X be a topological space and S a subset of X. The following statements are equivalent.*

(i) S is dense.

(ii) The only closed subset of X that contains S is X itself.

(iii) Every non-empty open subset of X contains at least one element of S.

Proof. Assume (i). S is dense in X. Hence $\bar{S} = X$. Let C be a closed subset of X such that $S \subset C$. From the definition of closure this means that it is the intersection all such closed sets C which contain S. Since it is all of X this happens when X only intersects itself. Therefore X is the only closed subset of X that contains S. Hence (i) implies (ii).

Assume (ii). The only closed subset of X that contains S is X itself. Hence X is the intersection of all closed subsets that contain S. By definition S is dense. Therefore (ii) implies (i).

Assume (ii). The only closed subset of X that contains S is X itself. Let U be a non-empty open subset of X. We want to show that $U \cap S \neq \emptyset$. Assume $U \cap S = \emptyset$. Let $C = X \backslash U$. C is closed in X. Since $U \cap S = \emptyset, S$ is a subset of C. But our initial assumption was that the only closed subset of X that contains S is X itself. Therefore $C = X$. Hence U must be empty. This is a contradiction, as U was non-empty. Therefore $U \cap S \neq \emptyset$. Hence every non-empty open subset of X contains at least one element of S. Therefore (ii) implies (iii).

Assume (iii). Every non-empty open subset of X contains at least one element of S. Let C be a closed set of X that contains S. Let U be the complement of C. Then $U \cap S = \emptyset$. Since U is the complement of the closed subset C, U is open in X. By assumption every non-empty open subset of X contains at least one element of S therefore U must be empty. Hence $C = X$. Therefore (iii) implies (ii).
Hence (i), (ii) and (iii) are equivalent. □

Lemma 1.14. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Proof. Let $a, b \in \mathbb{R}$ with $a < b$. We can always find $r \in \mathbb{Q}$ such that $a < r < b$. [ABB, Page 22] Let $U \subset \mathbb{R}$ be an open non-empty subset. Let $x \in U$. Then we know all y for which $x - \epsilon < y < x + \epsilon$ lie in U and ϵ a positive real number. We would like to show that at least one y is an element of $\mathbb{Q}$. $x - \epsilon < x + \epsilon$. Let $a = x - \epsilon$ and $b = x + \epsilon$. Hence $a < b$. From our given statement we know there is at least one $y \in \mathbb{Q}$ between a and b. Hence we have found that every non empty open subset of $\mathbb{R}$ contains at least one element of $\mathbb{Q}$. By Lemma 1.13 $\mathbb{Q}$ is dense in $\mathbb{R}$. □

1.2 Subspace topology

Definition 1.15. Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is defined by intersecting all the open sets of X with Y. That is, a subset U of Y is open in the subspace if $U = O \cap Y$ for O an open subset of X.

Theorem 1.16. *The subspace topology is a topology.*

Proof. We need to prove that,

(i) The empty set $\emptyset$ and the whole space Y are open.

(ii) Any finite intersection of open sets is open.

(iii) The union of open sets is open.

Proof of (i). $\emptyset$ is open in X. $\emptyset = \emptyset \cap Y$. Hence we have found an open subset of X namely $\emptyset$ for which $\emptyset = \emptyset \cap Y$. Hence $\emptyset$ is open in Y. X is open in X and $Y = X \cap Y$. Hence we have found an open subset of X namely X for which $Y = X \cap Y$. Therefore Y is open in Y.

Proof of (ii) Let $U_1 = O_1 \cap Y$ and $U_2 = O_2 \cap Y$ be open sets in Y for O_1, O_2 open subsets of X. Then $U_1 \cap U_2 = (O_1 \cap Y) \cap (O_2 \cap Y) = (O_1 \cap O_2) \cap Y$.

Since O_1, O_2 are open in X and X is a topological space $O_1 \cap O_2$ is open. Hence $U_1 \cap U_2$ is open in Y. This can be done for any finite intersection of open sets of Y. $(O_1 \cap Y) \cap (O_2 \cap Y) \cap \cdots \cap (O_n \cap Y) = (O_1 \cap O_2 \cap \cdots \cap O_n) \cap Y$.

Proof of (iii). Let U_i be open in Y. Then $U_i = O_i \cap Y$ for some open $O_i \subset X$. $\bigcup_{i \in I} (O_i \cap Y) = Y \cap \bigcup_{i \in I} O_i$. O_i is open in X and the union of open sets is open in X. Hence $Y \cap \bigcup_{i \in I} O_i$ is open in Y.

Therefore the subspace topology is a topology. □

Lemma 1.17. *Let X be a topological space and $Y \subset X$ with the subspace topology on Y. A subset Z of Y is closed in the subspace topology if and only if $Z = C \cap Y$ for C a closed subset of X.*

Proof. Assume Z is closed in Y. Then $Y \backslash Z$ is open in Y. In the subspace topology on Y, $Y \backslash Z = O \cap Y$ for O open in X. This gives us $X \backslash O$ is closed in X. $Z = Y \cap (X \backslash O)$. Hence Z is the intersection of a closed set of X with Y.
Assume $Z = C \cap Y$ with C closed in X. This gives $X \backslash C$ is open in X. By definition of subspace topology $(X \backslash C) \cap Y$ equals an open set in Y. But $(X \backslash C) \cap Y = Y \backslash Z$. Hence Z is closed. □

Lemma 1.18. *Let X be a topological space and $U \subseteq X$ an open subset. Let $W \subseteq U$ be open for the subspace topology on U. Then W is open in X.*

Proof. $W \subseteq U$ is open for the subspace topology on U. Hence $W = O \cap U$ for some open $O \subseteq X$. Hence W is the intersection of two open sets of X. Therefore W is open in X. □

Lemma 1.19. *Let X be a topological space and $V \subseteq X$ a closed subset. Let $Y \subseteq V$ be closed for the subspace topology on V. Then Y is closed in X.*

Proof. $Y \subseteq V$ is closed for the subspace topology on V. Hence $Y = C \cap V$ for some closed $C \subseteq X$. Hence Y is the intersection of two closed sets of X. Therefore Y is closed in X. □

Lemma 1.20. *Let X be a topological space and U an open subset of X. Let $S \subset X$ be a subset Then $\overline{S \cap U} = \bar{S} \cap U$ where $\overline{S \cap U}$ is closure in U and $\overline{S}$ is closure in X.*

Proof. $\subseteq$. By definition $\overline{S \cap U} = \bigcap \{K \supseteq S \cap U : K \text{ is closed in } U\}$. We want to show $\bar{S} \cap U$ is closed in U and contains $S \cap U$. i.e $\bar{S} \cap U$ is one these K in $\{K \supseteq S \cap U : K \text{ is closed in } U\}$. $\bar{S}$ is the closure of S in X and $\bar{S}$ is closed in X. By the subspace topology on U, $\bar{S} \cap U$ is closed in U. We have $\bar{S} \supseteq S$ and therefore $\bar{S} \cap U \supseteq S \cap U$. Hence $\bar{S} \cap U$ is indeed one of these K. Therefore $\overline{S \cap U} \subseteq \bar{S} \cap U$.

$\supseteq$ By definition $\bar{S} = \bigcap \{M \subseteq X : M \supseteq S \text{ and } M \text{ is closed in } X\}$. $\bar{S} \cap U = \bigcap (M \cap U)$. Let K be closed in U and contain $S \cap U$. We want to show for all K,

$K \supseteq \bigcap(M \cap U)$ or $K \supseteq M \cap U$ for some M. K can be expressed as $K = C \cap U$ for C closed in X. Put $M = C \cup (X \backslash U)$. $M = C \cup (X \backslash U)$ is closed in X by Lemma 1.8. We have $M \supseteq C \supseteq K \supseteq S \cap U$. Since $S \subseteq X$, $X \backslash U$ contains the points of S which are not in $S \cap U$. Therefore $C \cup (X \backslash U)$ contains all of S. $M \cap U = (C \cup (X \backslash U)) \cap U = (C \cap U) \cup ((X \backslash U) \cap U) = (C \cap U) \cup \emptyset = C \cap U = K$.Hence

$$\begin{aligned} K &\supseteq M \cap U \\ K &\supseteq \bigcap(M \cap U) \\ K &\supseteq \bar{S} \cap U \\ \bigcap K &\supseteq \bar{S} \cap U \\ \overline{S \cap U} &\supseteq \bar{S} \cap U \end{aligned}$$

Hence $\bar{S} \cap U = \overline{S \cap U}$. □

Lemma 1.21. *Let X be a topological space, $S \subseteq X$ dense and $\emptyset \neq V \subseteq X$ open. Then $V \cap S$ is dense in V.*

Proof. Let $\emptyset \neq U \subseteq V$ be open. By Lemma 1.18, U is open in X. $U \cap (V \cap S) = U \cap S$. Since U is open in X and S is dense in X then $U \cap S \neq \emptyset$. Therefore $U \cap (V \cap S) \neq \emptyset$ and hence $V \cap S$ is dense in V. □

Definition 1.22. Let X be a topological space. A *covering* C of X is a collection of subsets U_i of X whose union is all of X. We say C is an *open covering* if each U_i is in T the topology on X. i.e each U_i is an open set.

Let $X = \mathbb{R}^2$ with the standard topology. Let $U \subseteq \mathbb{R}^2$ and $P \in U$. We say U is a neighbourhood of P if there exists $\epsilon > 0$ such that for all Q, $|Q - P| < \epsilon$ we have that $Q \in U$.

Definition 1.23. An open neighbourhood of P is an open set $U \subseteq X$ with $P \in U$.

Definition 1.24. A neighbourhood of a point P is a subset $V \subseteq X$, such that $V \supseteq U$ for some open neighbourhood U of P.

Lemma 1.25. *Let X be a topological space covered by open subsets U_i.That is $X = \bigcup U_i$ for $i \in I$. Let $Z \subseteq X$ such that $Z \cap U_i$ is open in U_i for each $i \in I$. Then Z is open in X.*

Proof. $Z = \bigcup Z \cap U_i$. We are given that $Z \cap U_i$ is open in U_i. By Lemma 1.18 $Z \cap U_i$ is open in X. By definition of topology the union of all $Z \cap U_i$ is open in X. But $Z = \bigcup Z \cap U_i$, therefore Z is open in X. □

Lemma 1.26. *Let X be a topological space covered by open subsets U_i.That is $X = \bigcup U_i$ for $i \in I$. Let $Z \subseteq X$ such that $Z \cap U_i$ is closed in U_i for each $i \in I$. Then Z is closed in X.*

Proof. Let $X = \bigcup U_i$, $Z \subseteq X$ and $Z \cap U_i$ closed in U_i for each $i \in I$. Put $W = X \backslash Z$. $W \subseteq X$. Since $Z \cap U_i$ is closed in U_i, $W \cap U_i \subseteq U_i$ is open in U_i. By Lemma 1.25 W is open in X. But $W = X \backslash Z$. Hence Z is closed in X.

□

Lemma 1.27. *The Subspace Topology of the x-axis in $\mathbb{R}^2$ is simply the standard topology of $\mathbb{R}$.*

Proof. Let U be an open set in the subspace topology. This means that $U = O \cap x$-axis for some open subset $O \subset \mathbb{R}^2$. We need to show that each U is open in the standard topology of $\mathbb{R}$. From Lemma 1.5 we know that O is simply the union of open disks D_i that lie in O. If we intersect one of these open disks with the x-axis either this intersection is empty or it is an open interval from $(x_1, 0)$ to $(x_2, 0)$ for $x_1, x_2 \in \mathbb{R}$. Then this open interval which we can call (p, q) is open in the standard topology of $\mathbb{R}$ since for any $x \in (p, q)$ we can find an open interval (a, b) for which $x \in (a, b)$ that is contained in (p, q) namely (p, q). $\bigcup D_i \cap x$-axis is the union of open intervals in $\mathbb{R}$. Since each interval is open in $\mathbb{R}$, the union of these intervals is open.
Let (a, b) be an open interval in $\mathbb{R}$ with the standard topology. We construct an open disc D through (a, b) with centre $P = \frac{a+b}{2}$ and let $R \in D$ with $|RP| < \frac{b-a}{2}$ where $\frac{b-a}{2}$ is the radius of the disc D. Put $\epsilon = \frac{b-a}{2} - PR$. We want to show for all points Q in the small disc with centre R and radius ϵ,Q is also in D. i.e $|PQ| < \frac{b-a}{2}$. We have by the triangle inequality that $|PQ| \leq |PR| + |RQ|$. Since $|RQ| < \epsilon$,$|PQ| < |PR| + \epsilon$. $|PQ| < |PR| + \frac{b-a}{2} - |PR|$ so $|PQ| < \frac{b-a}{2}$. Hence Q lies in D. Hence D is open in $\mathbb{R}^2$. Therefore x-axis $\cap D$ is open in the subspace topology. Let $U \subseteq \mathbb{R}$ be an open subset with the standard topology in $\mathbb{R}$. U is the union of open intervals. That is, $U = \bigcup I_j$ where I_j is an open interval. Put $I_j = D_j \cap x$-axis where D_j is an open disc. Hence $U = (\bigcup D_j) \cap x$- axis. Since D_j is open in $\mathbb{R}^2$ the union of these open discs is open in $\mathbb{R}^2$. Hence U is open in the subspace topology. □

Definition 1.28. Let X be a topological space. X is defined as Hausdorff if the following holds. For any two points P, Q, where $P \neq Q$, there exist open subsets U and V in X such that $P \in U$ and $Q \in V$ and $U \cap V = \emptyset$.

$\mathbb{R}$ and $\mathbb{R}^2$ are both Hausdorff topological spaces with their respective standard topologies.

1.3 Mappings

Definition 1.29. Let X and Y be topological spaces. A map $f : X \to Y$ is *continuous* if for every open subset $V \subset Y$, $f^{-1}(V) \subset X$ is open in X. Note $f^{-1}(V)$ is the inverse image of V under the map f.

Lemma 1.30. *Let X and Y be topological spaces. Let $g : X \to Y$ be a continuous map. Let $C \subset Y$ be closed. Then $g^{-1}(C) \subset X$ is closed in X where g^{-1} is the inverse image.*

Proof. Let $C \subset Y$ be closed. Then $Y \backslash C$ is open in Y. Since g is continuous $g^{-1}(Y \backslash C)$ is open in X. Hence $X \backslash g^{-1}(Y \backslash C)$ is closed in X. But $X \backslash g^{-1}(Y \backslash C) = g^{-1}(C)$. Hence $g^{-1}(C)$ is closed in X. □

Lemma 1.31. *Let $\phi : A \to B$ and $\psi : B \to C$ be maps. Let $V \subseteq C$ be a subset. Then $\phi^{-1}(\psi^{-1}(V)) = (\psi \circ \phi)^{-1}(V)$. Note here that ϕ^{-1}, ψ^{-1} and $(\psi \circ \phi)^{-1}$ mean inverse image.*

Proof. $\phi^{-1}(\psi^{-1}(V)) = \{Q \in A : \phi(Q) \in \psi^{-1}(V)\} = \{Q \in A : \psi(\phi(Q)) \in V\}$.
We see $\{Q \in A : \psi(\phi(Q)) \in V\}$ is exactly the set $(\psi \circ \phi)^{-1}(V)$.
Hence $\phi^{-1}(\psi^{-1}(V)) = (\psi \circ \phi)^{-1}(V)$. □

Lemma 1.32. *Let $\phi : A \to B$ and $\psi : B \to C$ be continuous maps. Then the composition map $\psi \circ \phi : A \to C$ is continuous.*

Proof. Let $U \subseteq C$ be an open subset. Since ψ is continuous $\psi^{-1}(U)$ is open in B. Since ϕ is continuous $\phi^{-1}(\psi^{-1}(U))$ is open in A. By Lemma 1.31 $\phi^{-1}(\psi^{-1}(U)) = (\psi \circ \phi)^{-1}(U)$. Hence $(\psi \circ \phi)^{-1}(U)$ is open in A. Therefore $\psi \circ \phi : A \to C$ is continuous. □

Definition 1.33. Let $f : X \to Y$ be a map and $S \subseteq X$ a subset. The map,

$$\begin{aligned} f_{|S} : S &\to Y \\ s &\mapsto f(s) \end{aligned}$$

is called the restriction of f to S.

Lemma 1.34. *Let X, Y be topological spaces. Let $X = \bigcup_i U_i$ an open covering. Let $\phi : X \to Y$ be a map such that $\phi_{|U_i} : U_i \to Y$ is continuous for each i. Then ϕ is continuous.*

Proof. Let $V \subseteq Y$ be an open subset. Since $\phi_{|U_i}$ is continuous, $\phi^{-1}_{|U_i}(V)$ is open in U_i. By Lemma 1.18 $\phi^{-1}_{|U_i}(V)$ is open in X.

$$\begin{aligned} \phi^{-1}_{|U_i}(V) &= \{P \in U_i : \phi_{|U_i}(P) \in V\} \\ &= \{P \in U_i : \phi(P) \in V\} \end{aligned}$$

$$\begin{aligned} \phi^{-1}(V) &= \{P \in X : \phi(P) \in V\} \\ &= \{P \in \bigcup_i U_i : \phi(P) \in V\} \end{aligned}$$

We see $\phi^{-1}(V) = \bigcup_i \phi^{-1}_{|U_i}(V)$. Since each $\phi^{-1}_{|U_i}(V)$ is open in X, $\bigcup_i \phi^{-1}_{|U_i}(V)$ is open in X. Hence $\phi^{-1}(V)$ is open in X, proving ϕ is continuous.

□

Lemma 1.35. *Let $f : A \to B$ be a bijective map. Let $X \subseteq A$ and $Y \subseteq B$. Then $f^{-1}(f(X)) = X$ and $f(f^{-1}(Y)) = Y$. Note here f^{-1} means inverse image.*

Proof. $f^{-1}(f(X)) = \{P \in A : f(P) \in f(X)\}$.Hence $f^{-1}(f(X))$ must contain X. Therefore $X \subseteq f^{-1}(f(X))$. Let $Q \in f^{-1}(f(X))$. Then $f(Q) \in f(X)$. Hence there exists $Q' \in X$ such that $f(Q) = f(Q')$. But f is injective. Therefore $Q = Q'$. Hence $f^{-1}(f(X)) \subseteq X$. Therefore $f^{-1}(f(X)) = X$.

Let $P \in f(f^{-1}(Y))$. Then there exists a point $Q \in f^{-1}(Y)$ such that $f(Q) = P$. By definition $f^{-1}(Y) = \{R \in A : f(R) \in Y\}$. Hence $f(Q) = P \in Y$. Therefore $f(f^{-1}(Y)) \subseteq Y$. Let $P \in Y$. Since f is surjective there exists $Q \in A$ such that $f(Q) = P$. Hence $Q \in f^{-1}(Y)$. Therefore $f(Q) \in f(f^{-1}(Y))$. Hence $Y \subseteq f(f^{-1}(Y))$. Hence $Y = f(f^{-1}(Y))$. □

Definition 1.36. A map $f : X \to Y$ is *closed* if it maps closed sets in X to closed sets in Y.

Lemma 1.37. *Let $f : X \to Y$ be a bijective map. f^{-1} is continuous if and only if f is closed. Note here f^{-1} means inverse map of f.*

Proof. $\Rightarrow$ Assume f^{-1} is continuous. Since f^{-1} is continuous this means for every closed set $C \subset X$, $(f^{-1})^{-1}(C) \subset Y$ is closed. But $(f^{-1})^{-1}(C) = f(C)$ so $f(C)$ is closed. Hence f is closed.

$\Leftarrow$ Assume f is closed. This means f takes closed sets in X to closed sets in Y. Let C be a closed set in X. Then we need to show $(f^{-1})^{-1}(C)$ is closed in Y. f is a bijective map, therefore $(f^{-1})^{-1}(C) = f(C)$. f is closed by assumption so $f(C)$ is closed in Y. Therefore f^{-1} is continuous.

□

Definition 1.38. Let A, B be sets such that $A \subseteq B$. The *inclusion map* i is defined as,

$$i : A \to B$$

$$x \mapsto i(x) = x$$

Lemma 1.39. *Let X, Y be topological spaces and $Y' \subseteq Y$ a subset. Let $i : Y' \to Y$ be the inclusion map. Let $\phi : X \to Y'$ be a map such that $i \circ \phi$ is continuous. Then ϕ is continuous.*

Proof. Let $U \subseteq Y'$ be an open subset. By the subspace topology on Y', $U =$

$O \cap Y'$ for some open $O \subseteq Y$. $(i \circ \phi)^{-1}(O)$ is open in X.

$$\begin{aligned}(i \circ \phi)^{-1}(O) &= \{P \in X : (i \circ \phi)(P) \in O\} \\ &= \{P \in X : i(\phi(P)) \in O\} \\ &= \{P \in X : \phi(P) \in O \cap Y'\} \\ &= \{P \in X : \phi(P) \in U\} \\ &= \phi^{-1}(U)\end{aligned}$$

Hence $\phi^{-1}(U)$ is open in X and ϕ is continuous. □

Definition 1.40. A function f between two topological spaces is a *homeomorphism* if the following properties hold. f is a bijection, f is continuous and the inverse function f^{-1} is continuous.

Lemma 1.41. *Let $\phi : X \to Y$ be a homeomorphism and $S \subset X$ a subset. Then $\phi(\overline{S}) = \overline{\phi(S)}$.*

Proof. By definition $\overline{\phi(S)} = \bigcap\{D \supseteq \phi(S) : D \text{ is closed in } Y\}$.
$\overline{S} \supseteq S$ hence $\phi(\overline{S}) \supseteq \phi(S)$. Since $\overline{S}$ is closed in X, $\phi(\overline{S})$ is closed in Y. Hence $\phi(\overline{S}) \supseteq \overline{\phi(S)}$.

$\overline{\phi(S)} = \bigcap\{D \supseteq \phi(S) : D \text{ is closed in } Y\}$. We pick arbitrary D and show $D \supseteq \phi(\overline{S})$. $\overline{S} = \bigcap\{C \supseteq S : C \text{ is closed in } X\}$. We have that $D \supseteq \phi(S)$. Then $\phi^{-1}(D) \supseteq \phi^{-1}\phi(S)$ from which we obtain $\phi^{-1}(D) \supseteq S$. $\phi^{-1}(D)$ is closed in X by homeomorphism. $\phi^{-1}(D)$ is one such C in the closure of S and hence $\phi^{-1}(D) \supseteq \overline{S}$. Taking ϕ of both sides we get $D \supseteq \phi(\overline{S})$. $D \supseteq \phi(\overline{S})$ means $\bigcap(D) \supseteq \phi(\overline{S})$ and hence $\overline{\phi(S)} \supseteq \phi(\overline{S})$.
Therefore $\phi(\overline{S}) = \overline{\phi(S)}$. □

1.4 Irreducible topological spaces

Definition 1.42. A topological space X is irreducible if it cannot be expressed as the union of two proper closed subsets. A proper closed subset of X is a closed subset of X that is not all of X.

Theorem 1.43. *$\mathbb{R}$ is not an irreducible topological space.*

Proof. Let $A = \{x \leq 0 | x \in \mathbb{R}\}$. A^c is open in the standard topology of $\mathbb{R}$ since for any $y \in \mathbb{R}$ and $y > 0$ we can find an open interval (a, b) that contains y namely $(0, y + 1)$. Hence A is closed in $\mathbb{R}$ and $A \neq \mathbb{R}$.
Let $B = \{x \geq 0 | x \in \mathbb{R}\}$. B^c is open in the standard topology of $\mathbb{R}$ since for any $y \in \mathbb{R}$ and $y < 0$ we can find an open interval (p, q) that contains y namely $(y - 1, 0)$. Hence B is closed in $\mathbb{R}$ and $B \neq \mathbb{R}$. We see $\mathbb{R}$ can be expressed as the union of two proper closed subsets of $\mathbb{R}$. Therefore $\mathbb{R}$ is not an irreducible topological space. □

Definition 1.44. A non-empty subset Y of a topological space X is irreducible if it is irreducible as a topological space for the subspace topology.

Lemma 1.45. *Let X be an irreducible topological space. Let U, V be two non-empty open subsets of X. Then $U \cap V$ is a non-empty open subset.*

Proof. $U \cap V$ is open by definition of a topology. Assume $U \cap V = \emptyset$. Let $U' = X - U$ and $V' = X - V$. Then U', V' are closed subsets of X which are proper. Since $U \cap V = \emptyset$, $U' \cup V' = X$. X is therefore not irreducible, which is a contradiction. Therefore $U \cap V \neq \emptyset$. □

Theorem 1.46. *Let U be a non-empty open set of an irreducible topological space X. Then U is irreducible and dense in the subspace topology.*

Proof. Let $U \subset X$ be a non-empty open subset. Let $V \subset X$ be a non-empty open subset. By Lemma 1.45 $U \cap V$ is a non-empty open subset. $U \cap V$ contains at least one element of U. Hence by Lemma 1.13 U is dense.
Let $U \subset X$ be a non-empty open subset and suppose U is not irreducible. Then $U = U_1 \cup U_2$ for U_1, U_2 two proper closed subsets of U. In the subspace topology $U_1 = C_1 \cap U$ and $U_2 = C_2 \cap U$ for C_1, C_2 closed subsets in X. $C_1 \cup C_2$ contains U and U^c is closed in X. Therefore $C_1 \cup C_2 \cup U^c$ is all of X. But we have that X is irreducible so either $C_1 \cup C_2 = X$ or $U^c = X$. If $C_1 \cup C_2 = X$ then, given that X is irreducible either $C_1 = X$ or $C_2 = X$. If $C_1 = X$ then $U_1 = X \cap U$ which means $U_1 = U$ a contradiction since U_1 is a proper subset of U. If $C_2 = X$ then $U_2 = X \cap U$ which means $U_2 = U$. Again this is a contradiction since U_2 is a proper subset of U. If $U^c = X$ then U is empty. This is a contradiction since we assumed U is a non-empty set. Hence U must be irreducible. □

Lemma 1.47. *If Y is an irreducible subset of a topological space X then $\overline{Y}$ is irreducible in the subspace topology.*

Proof. Let $Y \subset X$ be irreducible in the subspace topology. Suppose $\overline{Y}$ is not irreducible. Then $\overline{Y} = Y_1 \cup Y_2$ for Y_1, Y_2 proper closed subsets of $\overline{Y}$. In the subspace topology on $\overline{Y}$, $Y_1 = C_1 \cap \overline{Y}$ and $Y_2 = C_2 \cap \overline{Y}$ with C_1, C_2 closed in X. So we have $\overline{Y} = (C_1 \cap \overline{Y}) \cup (C_2 \cap \overline{Y}) = \overline{Y} \cap (C_1 \cup C_2)$ which implies $\overline{Y} \subseteq C_1 \cup C_2$. Since $Y \subseteq \overline{Y}$ then

$$
\begin{aligned}
Y &= Y \cap \overline{Y} \\
Y &= Y \cap (Y_1 \cup Y_2) \\
Y &= (Y \cap Y_1) \cup (Y \cap Y_2) \\
Y &= (Y \cap (C_1 \cap \overline{Y})) \cup (Y \cap (C_2 \cap \overline{Y})) \\
Y &= (Y \cap C_1) \cup (Y \cap C_2)
\end{aligned}
$$

By the subspace topology on Y, $Y \cap C_1$ and $Y \cap C_2$ are closed in Y. Since Y is irreducible either $Y = Y \cap C_1$ or $Y = Y \cap C_2$. If $Y = Y \cap C_1$ this implies $Y \subset C_1$. Since the closure of Y is the intersection of all closed sets that contain Y then

$\overline{Y} \subset C_1$. But $Y_1 = C_1 \cap \overline{Y}$ so we have $Y_1 = \overline{Y}$ which is a contradiction since Y_1 was a proper subset of $\overline{Y}$. If $Y = Y \cap C_2$ this implies $Y \subset C_2$ and hence $\overline{Y} \subset C_2$ but $Y_2 = C_2 \cap \overline{Y}$ which implies $Y_2 = \overline{Y}$. This again is a contradiction. □

Lemma 1.48. *Let X, Y be topological spaces with X irreducible. Let $\varphi : X \to Y$ be a continuous surjective map. Then Y is irreducible.*

Proof. Assume Y is not irreducible. i.e $Y = Y_1 \cup Y_2$ for $Y_1 \subsetneq Y$, $Y_2 \subsetneq Y$, and Y_1, Y_2 closed in Y. $\varphi^{-1}(Y) = \varphi^{-1}(Y_1 \cup Y_2) = \varphi^{-1}(Y_1) \cup \varphi^{-1}(Y_2)$. φ is continuous therefore $\varphi^{-1}(Y_1), \varphi^{-1}(Y_2)$ are closed in X. $X = \varphi^{-1}(Y_1) \cup \varphi^{-1}(Y_2)$. By assumption X is irreducible hence $X = \varphi^{-1}(Y_1)$ or $X = \varphi^{-1}(Y_2)$. wlog we can assume $X = \varphi^{-1}(Y_1)$. This means $\varphi(X) \subseteq Y_1$. This is a contradiction since φ is surjective. Hence Y is irreducible. □

Definition 1.49. Let S be a set. A *minimal element* in S is defined as an element of S that is not larger than any other element in S.

Definition 1.50. Let S be a set. A *maximal element* in S is defined as an element of S that is not smaller than any other element in S.

Lemma 1.51. *Let X be a topological space. The following statements are equivalent.*

- *i X is noetherian.*
- *ii Every non-empty collection of closed subsets contains a minimal element.*
- *iii X satisfies the ascending chain condition for open subsets.*
- *iv Every non-empty collection of open subsets contains a maximal element.*

Proof. Assume (i). X is a noetherian topological space. Let S be a non-empty collection of closed subsets of X which does not contain a minimal element. Then for $Y_1 \in S$ there is $Y_2 \in S$ such that $Y_1 \supsetneq Y_2$. There exists $Y_3 \in S$ such that $Y_2 \supsetneq Y_3$. Since S contains no minimal element this chain will continue on. Hence we have a contradiction to X being noetherian. Therefore (i) implies (ii).

Assume (i). X is a noetherian topological space. Let S be an ascending chain of open subsets $O_1 \subseteq O_2 \subseteq \ldots$ which does not become stationary. Taking the complements of these open subsets we obtain a chain of descending closed subsets $O_1^c \supseteq O_2^c \supseteq \ldots$.Hence we have a contradiction to the assumption. Therefore (i) implies (iii).

Assume (iii) and let T be a descending chain of closed subsets $C_1 \supseteq C_2 \supseteq \ldots$ which does not become stationary. Taking the complements of these closed subsets we obtain a chain of ascending open subsets $C_1^c \subseteq C_2^c \subseteq \ldots$.Hence we have a contradiction to the assumption. Therefore (iii) implies (i).

Assume (ii). Let S be a non-empty collection of open subsets of X which does not contain a maximal element. Let S' be the collection of the complements of the open sets of S. Then we obtain a non-empty set S' of closed subsets which does not contain a minimal element. Hence a contradiction to the assumption. Therefore (ii) implies (iv).

Assume (iv) Let T be a non empty collection of closed subsets of X which does not contain a minimal element. Let T' be the collection of the complements of the closed sets of T. Then we obtain a non empty set T' of open subsets which does not contain a maximal element. Hence a contradiction to the assumption. Therefore (iv) implies (ii).

Assume (ii). Suppose X is not noetherian. Then there is a chain of descending closed subsets of X which does not become stationary say $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_r \supseteq \ldots$. But this is a collection of closed subsets and we have assumed every non empty collection of closed subsets contains a minimal element. No element of this chain is minimal therefore we have a contradiction and X must be noetherian, Hence (ii) implies (i).

□

Lemma 1.52. *Let $Y = Y_1 \cup \cdots \cup Y_r$ be a non-empty closed subset of a topological space X with each Y_i an irreducible closed subset of Y and $Y_i \not\supseteq Y_j$ for $i \neq j$. Then $\overline{Y - Y_1} = Y_2 \cup Y_3 \cup \cdots \cup Y_r$ where $\overline{Y - Y_1}$ is closure in Y.*

Proof. $\overline{Y - Y_1}$ is the intersection of all closed subsets of Y that contains $Y - Y_1$. One such closed subset is $Y_2 \cup Y_3 \cup \cdots \cup Y_r$. We need to prove there is no smaller closed set than this containing $Y - Y_1$.

$(\overline{Y - Y_1}) \cap Y_i$ is closed in Y_i for the subspace topology on Y_i. $(Y - Y_1) \cap Y_i$ is open for the subspace topology on Y_i. But $(Y - Y_1) \cap Y_i = Y_i - Y_1$. $Y_i - Y_1$ is non-empty and open in Y_i. $Y_i - Y_1$ is dense in Y_i and the closure of $Y_i - Y_1$ is all of Y_i. Therefore $\overline{(Y - Y_1)} \cap Y_i = Y_i$. Hence $\overline{Y - Y_1} \supseteq Y_i$. Since this is true for each Y_i up to r, $\overline{Y - Y_1} \supseteq Y_2 \cup Y_3 \ldots Y_r$.
We have $Y - Y_1 = Y_2 \cap Y_3 \cup \cdots \cup Y_r$. By definition of closure, $\overline{Y - Y_1} \subseteq Y_2 \cup \cdots \cup Y_r$.
Therefore $\overline{Y - Y_1} = Y_2 \cup Y_3 \ldots Y_r$ □

Theorem 1.53. *In a noetherian topological space X every non-empty subset Y can be expressed as a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets Y_i. If $Y_i \not\supseteq Y_j$ for $i \neq j$ then Y_i are uniquely determined and are called the* irreducible components *of Y.*

Proof. Let C be the set of non-empty closed subsets which cannot be written as a finite union of irreducible closed subsets. We will show C is in fact empty. Assume C is non-empty. Since X is noetherian, C contains a minimal element Y. Then by construction of C, Y is reducible. Therefore $Y = C_1 \cup C_2$ with C_1 and C_2 proper closed subsets of Y. Since Y is minimal then C_1 and C_2 are not elements of C. Hence C_1 and C_2 can be written as a finite union of irreducible closed subsets. Therefore, so can Y, which is a contradiction. Hence $C = \emptyset$.
To show uniqueness we use induction on r.

Our statement $P(r)$: If $Y \subseteq X$ is non-empty closed such that $Y = Y_1 \cup \ldots Y_r$ with Y_i irreducible closed and $Y_i \not\supseteq Y_j$ for $i \neq j$ and $Y = Y_1' \cup \ldots Y_s'$ is another such representation of Y with Y_i' irreducible closed and $Y_i' \not\supseteq Y_j'$ for $i \neq j$ then $s = r$ and $Y_1', \ldots Y_s'$ is just a renumbering of $Y_1, \ldots Y_r$.

$P(1)$: This is trivial since when $r = 1$ we have $Y = Y_1$ with Y_1 irreducible closed. This means in our other representation we must have that $s = 1$ also. Hence for $P(1)$, $s = 1 = r$.

We assume that $P(r-1)$ is true.

We now show $P(r)$ is true. Assume $Y = Y_1 \cup \dots Y_r = Y = Y'_1 \cup \dots Y'_s$ as in $P(r)$. By this assumption $Y'_1 \subseteq Y = Y_1 \cup \dots \cup Y_r$ so $Y'_1 = \bigcup (Y'_1 \cap Y_i)$. But Y'_1 is irreducible, so $Y'_1 \subseteq Y_i$ for some i say $i = 1$. Similarly $Y_1 \subseteq Y'_j$ for some j. Then $Y'_1 \subseteq Y'_j$ so by assumption $j = 1$. Hence $Y_1 = Y'_1$.

Put $Z = \overline{Y - Y_1}$. From Lemma 1.52 $Z = \overline{Y - Y_1} = Y_2 \cup Y_3 \dots Y_r$. Also $Z = Y'_2 \cup \dots \cup Y'_s$ is another representation. By assumption $P(r-1)$ is true. Hence $r - 1 = s - 1$ and $Y'_2, \dots Y'_s$ is just a renumbering of $Y_2, \dots Y_r$. Hence $s = r$ and $Y'_1, \dots Y'_s$ is just a renumbering of $Y_1, \dots Y_r$.
This shows the Y_i are uniquely determined.

Hence $P(r)$ is true.

□

Lemma 1.54. *Let Y be a topological space, $S \subseteq Y$ dense and $\emptyset \neq V \subseteq Y$ open. Then $V \cap S$ is dense in V.*

Proof. Let $\emptyset \neq U \subseteq V$ be open. $U \cap (V \cap S) = U \cap S$. Since U is open in Y and S is dense in Y then $U \cap S \neq \emptyset$ by Lemma 1.13. Therefore $U \cap (V \cap S) \neq \emptyset$ and hence $V \cap S$ is dense in V. □

Lemma 1.55. *Let $\varphi : X \to Y$ be a map. Let $V \subseteq Y$.Then $\varphi(\varphi^{-1}(V)) = V \cap \varphi(X)$*

Proof. Let $y \in V \cap \varphi(X)$. Then $y \in V$ and $y \in \varphi(X)$. Since $y \in \varphi(X)$, $y = \varphi(P)$ for some $P \in X$. Also $y \in V$, therefore $\varphi(P) \in V$. Hence $P \in \varphi^{-1}(V)$. Therefore $y = \varphi(P) \in \varphi(\varphi^{-1}(V))$. Hence $V \cap \varphi(X) \subseteq \varphi(\varphi^{-1}(V))$.

Let $y \in \varphi(\varphi^{-1}(V))$. Then $y = \varphi(Q)$ for some $Q \in \varphi^{-1}(V)$. Hence $\varphi(Q) \in V$. Therefore $y \in V$. Since $\varphi^{-1}(V) \subseteq X$. $\varphi(Q) \in \varphi(X)$. Hence $y \in \varphi(X)$. Therefore $y \in V \cap \varphi(X)$. $\varphi(\varphi^{-1}(V)) \subseteq V \cap \varphi(X)$. Hence $\varphi(\varphi^{-1}(V)) = V \cap \varphi(X)$. □

Definition 1.56. We define the *dimension* of a topological space X denoted dim$X$ to be the supremum of all $n \in \mathbb{Z}$ such that there exists a chain $Z_0 \subset Z_1 \subset \dots Z_n$ of distinct irreducible closed subsets of X.

Given the following chains in X, $Z_0 \subset Z_1 \subset Z_2$, $T_0 \subset T_1 \subset T_2 \subset T_3$ and $S_0 \subset S_1 \subset S_2 \subset S_3 \subset S_4$ the dimension of X is 4. The dim X is the length of the longest chain of irreducible closed subsets of X.

Lemma 1.57. *If Y is a subset of a topological space X, then dim $Y \leq$ dim$X$*

Proof. Let $C = Z_0 \subsetneq Z_1 \subsetneq \dots \subsetneq Z_n$ be a chain of irreducible closed subsets of Y such that dim $Y = n$. Then for the subspace topology on Y, $Z_i = X_i \cap Y$ for

X_i closed in X. $C = X_0 \cap Y \subsetneq X_1 \cap Y \subsetneq \cdots \subsetneq X_n \cap Y$. Hence we have a chain $\overline{Z_0} \subsetneq \overline{Z_1} \subsetneq \cdots \subsetneq \overline{Z_n}$ of closed subsets of X with $\overline{Z_i}$ irreducible by Lemma 1.47. They are distinct since if they were not then $Z_i = Y \cap \overline{Z_i} = Y \cap \overline{Z_{i+1}} = Z_{i+1}$ which is a contradiction $\square$

2 Affine Space

This section will look at the affine space with particular interest in affine varieties. It is important to discuss the polynomial ring and some important properties in advance. Also we note the following,

Definition 2.1. Let K be a field. We say K is *algebraically closed* if for non-constant $f \in K[t]$, f has a root in K.

Lemma 2.2. *If K is an algebraically closed field, then K is infinite.*

Proof. Suppose that K is finite. Say $K = \{a_1, a_2, \ldots, a_n\}$. Let $f(x)$ be the polynomial $f(x) = (x-a_1)(x-a_2)\ldots(x-a_n)+1 \in K[x]$. Then $f(x)$ has no roots in K since $f(x) \neq 0$ for any elements in K. Hence K is not algebraically closed. Therefore, we can conclude that any algebraically closed field is infinite. □

Definition 2.3. [AM, Page 80] A ring R is Noetherian if it satisfies one of the following equivalent statements.

(i) the ascending chain condition on ideals of R. This means given an increasing sequence of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \cdots \subseteq I_{k-1} \subseteq I_k \subseteq I_{k+1} \ldots$$

there exists $n \in \mathbb{N}$ such that $I_n = I_{n+1} = \ldots$

(ii) Every ideal in R is finitely generated.

Lemma 2.4. *A field K is Noetherian*

Proof. A field K has only two ideals. The 0 ideal and the field itself. Hence we have an ascending chain of ideals $0 \subseteq 0 \subseteq \cdots \subseteq K \subseteq K$. We can find an n such that $I_n = I_{n+1}$. □

Unless stated otherwise we assume K is an algebraically closed field. Let $A = K[x_1, x_2, \ldots x_n]$ be the polynomial ring in n variables over a field K.

Lemma 2.5. $A = K[x_1, x_2, \ldots, x_n]$ *is a Noetherian ring.*

Proof. If R is Noetherian then $R[x_1, \ldots x_n]$ is Noetherian [AM, corollary 7.6 page 81] . Here we have the polynomial ring $A = K[x_1, \ldots, x_n]$ over a field K. By Lemma 2.4 K is Noetherian. Hence the polynomial ring over K is also Noetherian. □

Definition 2.6. An integral domain is a non-zero commutative ring R in which if $ab = 0$ for $a, b \in R$ then either $a = 0$ or $b = 0$.

Theorem 2.7. *If a ring R is an integral domain then $R[t]$ is an integral domain.*

Proof. Let f and g be non-zero polynomials in $R[t]$. i.e $f = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t^1 + a_0$ and $g = b_mt^m + b_{m-1}t^{m-1} + \cdots + b_1t^1 + b_0$ with $a_i, b_k \in R$ and $a_n, b_m \neq 0$. Then $fg = a_nb_mt^{n+m} + \cdots + a_0b_0$. Considering the first term $a_nb_mt^{n+m}$, $a_nb_m \neq 0$ since R is an integral domain then $fg \neq 0$ for f and g non zero. □

Theorem 2.8. *If a ring R is an integral domain then $R[t_1, \dots, t_n]$ is an integral domain.*

Proof. We prove this lemma using induction on n. Our statement $P(n)$ is if a ring R is an integral domain then $R[t_1, \dots, t_n]$ is an integral domain. In Theorem 2.7 we proved the base case for $n = 1$. If R is an integral domain then $R[t]$ is an integral domain. Let $n = 2$. Since $R = R[t_1]$ is now an integral domain, Theorem 2.7 tells us $R[t_1][t_2]$ is an integral domain. This ring is isomorphic to $R[t_1, t_2]$ and therefore $R[t_1, t_2]$ is an integral domain. Assume our statement is true for a fixed $k \geq 1$. i.e. For R an integral domain then $R[t_1 \dots t_k]$ is an integral domain. We must prove $P(k + 1)$ is true. We have $R[t_1 \dots t_k]$ is an integral domain. By Theorem 2.7 we have that $R[t_1 \dots t_k][t_{k+1}]$ is an integral domain. $R[t_1 \dots t_k][t_{k+1}] \cong R[t_1, \dots, t_{k+1}]$. Hence if R is an integral domain then $R[t_1, \dots, t_n]$ is an integral domain. □

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From this we can state the following. Since a field is by definition an integral domain, the polynomial ring $A = K[x_1, \dots x_n]$ is an integral domain.

Definition 2.9. Let R be a ring. Let $r \in R$. r is a unit in R if there exists $s \in R$ such that $rs = 1$ where 1 is the multiplicative identity in R.

Lemma 2.10. *The units in the polynomial ring $K[t]$ are the constant polynomials.*

Proof. Assume $f(t)$ is a unit in $K[t]$. This means there exists $g(t) \in K[t]$ such that $f(t) \cdot g(t) = 1$. If f, g are polynomials with coefficients in K, then $\deg(fg) = \deg f + \deg g$.[SL, Theorem 1.4 page 112] Hence $\deg f(t) + \deg g(t) = \deg 1$. The polynomial 1 has degree 0. Hence $\deg f(t) + \deg g(t) = 0$. This can only happen when $\deg f(t) = \deg g(t) = 0$. Therefore $f(t)$ is a constant polynomial. Hence the units in $K[t]$ are the constant polynomials. □

Definition 2.11. Let R be a ring. An element $p \in R$ is irreducible if p is not a unit and given a factorization $p = ab$ with $a, b \in R$ then a or b is a unit.

Definition 2.12. A unique factorization domain(UFD) is a ring R that is an integral domain in which every non zero element which is not a unit, has a unique factorization into irreducible elements.

Definition 2.13. A polynomial $f \in K[t]$ is *irreducible* if it is of degree ≥ 1, and if, given a factorization $f = gh$ with $g, h \in K[t]$, then $\deg g = 0$ or $\deg h = 0$.. Therefore, up to a non-zero constant factor, the only divisors of f are f and 1.

Theorem 2.14. *A field K is a UFD.*

Proof. K is an integral domain. The non-zero elements in a field are all units. Hence by definition a field does not contain any non-zero, non-unit elements. Hence it is trivially a UFD. □

If R is a UFD then $R[t]$ is a UFD. [SL, Theorem 6.9 page 148] And again using induction we can prove that $R[t_1 \dots t_n]$ is a UFD. Since a field K is a UFD, $K[x_1 \dots x_n]$ is a UFD.

Definition 2.15. Let R be a commutative ring and I an ideal of R. The radical of I is defined as

$$\sqrt{I} = \{x \in R|\ x^n \in I \text{ for some } n > 0\} \tag{1}$$

Lemma 2.16. *$\sqrt{I}$ is itself an ideal.*

Proof. Let I be an ideal of a commutative ring R and $\sqrt{I}$ its radical. We need to show

(i) $0_R \in \sqrt{I}$ where 0_R is the zero element in R.

(ii) For $x \in \sqrt{I}$ and $r \in R$ then $rx \in \sqrt{I}$.

(iii) If $x, y \in \sqrt{I}$ then $x + y \in \sqrt{I}$.

Proof of (i). Let $0_R \in R$ be the zero element. $0_R \in I$ since I is an ideal. Let $n = 1$. $0_R^1 = 0_R$. Hence $0_R^1 \in I$. Therefore $0_R \in \sqrt{I}$.

Proof of (ii). Let $x \in \sqrt{I}$ and $r \in R$. Then by definition $x^n \in I$. $(rx)^n = r^n x^n$. Since I is an ideal and $x^n \in I$, $r^n \in R$ then $r^n x^n \in I$. Hence $rx \in \sqrt{I}$.

Proof of (iii). Let $x, y \in \sqrt{I}$. Then $x^n, y^m \in I$.

$$(x+y)^{n+m} = \sum_{i=0}^{n+m} \binom{n+m}{i} x^{n+m-i} y^i$$
$$(x+y)^{n+m} = \binom{n+m}{0} x^{n+m} + \binom{n+m}{1} x^{n+m-1} y^1 + \cdots + \binom{n+m}{n+m} y^{n+m}$$

$x^{n+m} \in I$ since $x^n \in I$. $y^{n+m} \in I$ since $y^m \in I$. If $i \leq m$ then $m - i \geq 0$ and $m+n-i \geq n$ hence $x^{n+m-i} \in I$. If $i \geq m$ then $y^i \in I$. So when $x^{n+m-i} \in R$ and $y^i \in I$ then $x^{n+m-i} y^i \in I$. When $x^{n+m-i} \in I$ and $y^i \in R$ then $x^{n+m-i} y^i \in I$. Therefore, $(x+y)^{n+m} \in I$. Hence $x + y \in \sqrt{I}$.

Hence $\sqrt{I}$ is an ideal. □

2.1 Affine n-space

Definition 2.17. The affine n-space over a field K denoted $\mathbf{A}^n$ is defined as the set of all n-tuples of elements of K.

$P \in \mathbf{A}^n$ is called a point, and if $P = (a_1, a_2, \dots a_n)$ with $a_i \in K$ then these a_i are the coordinates of P.

Theorem 2.18. *For T a subset of $A = K[x_1, \dots, x_n]$, the set $\mathfrak{a}$ of all elements*

$$g_1 f_1 + g_2 f_2 + \cdots + g_n f_n$$

such that n is any natural number, $g_i \in A, f_i \in T$ is an ideal.

Proof. Clearly this set of elements is a subset of A. We must prove the three properties for an ideal.

i The additive neutral element of A is the zero polynomial. For $n = 1$ we have $0_A f_1$ which is 0. Hence the additive neutral element is in $\mathfrak{a}$.

ii Let $g_1f_1 + g_2f_2 + \cdots + g_nf_n$ and $h_1f_{n+1} + h_2f_{n+2} + \cdots + h_rf_{n+r}$, $n, r \in \mathbb{N}$ be elements of $\mathfrak{a}$. When we add these elements we get $g_1f_1 + g_2f_2 + \cdots + g_nf_n + h_1f_{n+1} + h_2f_{n+2} + \cdots + h_rf_{n+r}$ which again is in $\mathfrak{a}$.

iii Let $g \in A$ we have $g(g_1f_1 + g_2f_2 + \ldots g_nf_n) = gg_1f_1 + gg_2f_2 + \ldots gg_nf_n$ where $gg_j \in A$.

If $T = \emptyset$ then T contains no f_i. In this case $\mathfrak{a}$ is the zero ideal.

Hence $\mathfrak{a}$ is indeed an ideal. □

Definition 2.19. The ideal $\mathfrak{a}$ in Theorem 2.18 is called the ideal in A *generated* by T.

2.1.1 Zero set of polynomials

Definition 2.20. Given T a subset of A, we define the zero set of T as

$$Z(T) = \{P \in \mathbf{A}^n | f(P) = 0 \text{ for all } f \in T\}$$

Lemma 2.21. *If $T_1 \subseteq T_2$ are subsets of A, then $Z(T_2) \subseteq Z(T_1)$.*

Proof. Let $P \in Z(T_2)$ then for all $f \in T_2$, $f(P) = 0$. Since $T_1 \subseteq T_2$, let $f \in T_1$, then $f(P) = 0$ hence $P \in Z(T_1)$. Therefore $Z(T_2) \subseteq Z(T_1)$. □

Theorem 2.22. *If $\mathfrak{a}$ is the ideal of A generated by T, then $Z(T) = Z(\mathfrak{a})$*

Proof. $\supseteq$. Since $T \subseteq \mathfrak{a}$, by Lemma 2.21 $Z(\mathfrak{a}) \subseteq Z(T)$.
$\subseteq$. Let $P \in Z(T)$. Take any element from $\mathfrak{a}$. Say $h = g_1f_1 + g_2f_2 + \cdots + g_nf_n$ where $f_i \in T$. Since $P \in Z(T)$, $f_1(P) = 0, f_2(P) = 0, \ldots f_n(P) = 0$.

$$\begin{aligned}
h(P) &= g_1(P)f_1(P) + g_2(P)f_2(P) + \cdots + g_n(P)f_n(P) \\
h(P) &= g_1(P)(0) + g_2(P)(0) + \cdots + g_n(P)(0) \\
h(P) &= 0 + 0 + \cdots + 0 \\
h(P) &= 0
\end{aligned}$$

Therefore $P \in Z(\mathfrak{a})$. Hence $Z(T) \subseteq Z(\mathfrak{a})$.
Since $Z(\mathfrak{a}) \subseteq Z(T)$ and $Z(T) \subseteq Z(\mathfrak{a})$ then $Z(T) = Z(\mathfrak{a})$ □

Lemma 2.23. *Given $T \subset A = K[x_1, \ldots x_n]$, $Z(T)$ can be expressed as the common zeros of a finite set of polynomials $f_1, f_2, \ldots f_r$.*

Proof. Let $T \subset A$. Let $\mathfrak{a}$ be the ideal generated by T. From Theorem 2.22 we have $Z(T) = Z(\mathfrak{a})$. Since A is a Noetherian ring, every ideal is finitely generated. Hence we have a finite set of polynomials which generate $\mathfrak{a}$. Let this set be denoted $T^{'}$. Theorem 2.22 tells us $Z(T^{'}) = Z(\mathfrak{a})$. Hence $Z(T) = Z(T^{'})$. Therefore $Z(T)$ can be expressed as the common zeros of a finite set $f_1, f_2 \ldots, f_r$. □

Definition 2.24. Let Y be a subset of $\mathbf{A}^n$. Y is an *algebraic set* if there exists a subset $T \subseteq A$ such that $Y = Z(T)$.

Lemma 2.25. *The union of two algebraic sets is algebraic.*

Proof. Let Y_1, Y_2 be algebraic sets. Then $Y_1 = Z(T_1)$ for some $T_1 \subset A$ and $Y_2 = Z(T_2)$ for some $T_2 \subset A$. $Y_1 \cup Y_2 = Z(T_1) \cup Z(T_2)$. We will show $Z(T_1) \cup Z(T_2) = Z(T_1T_2)$ where T_1T_2 is the set of all products of elements of T_1 by elements of T_2. i.e products of polynomials.

Let $P \in Y_1 \cup Y_2$. Then $P \in Y_1$ or $P \in Y_2$.

If $P \in Y_1$ then $f(P) = 0$ for all $f \in T_1$. Hence, we have $f(P) \cdot g(P) = 0 \cdot g(P)$. for all $g \in T_2$. $0 \cdot g(P) = 0$. Hence $P \in Z(T_1T_2)$.
If $P \in Y_2$ then $g(P) = 0$ for some $g \in T_2$. Hence we have $f(P) \cdot g(P) = f(P) \cdot 0$. for all $f \in T_1$. $f(P) \cdot 0 = 0$. Hence $P \in Z(T_1T_2)$.

Hence $Z(T_1 \cup T_2) \subseteq Z(T_1T_2)$.

Let $P \in Z(T_1T_2)$ such that $P \notin Y_1$. Then there exists $f \in T_1$, such that $f(P) \neq 0$. For all $g \in T_2$ we have $0 = (fg)(P) = f(P)g(P)$. Since $f(P) \neq 0$, we have $g(P) = 0$ for all $g \in T_2$. Hence $P \in Y_2$.
Let $P \in Z(T_1T_2)$ such that $P \notin Y_2$. Then there exists $g \in T_2$, such that $g(P) \neq 0$. For all $f \in T_1$ we have $0 = (fg)(P) = f(P)g(P)$. Since $g(P) \neq 0$, then $f(P) = 0$ for all $f \in T_1$. Hence $P \in Y_1$.
Therefore $Z(T_1T_2) \subseteq Z(T_1 \cup T_2)$.

Therefore $Z(T_1 \cup T_2) = Z(T_1T_2)$ showing the union of two algebraic sets is algebraic. □

Lemma 2.26. *The intersection of any family of algebraic sets is algebraic.*

Proof. Let $\{Y_\alpha\}$ be a set of algebraic sets over an index α. Then each $Y_\alpha = Z(T_\alpha)$ for $T_\alpha \subseteq A$. $\bigcap_\alpha Y_\alpha = \bigcap_\alpha Z(T_\alpha)$. We will show $\bigcap_\alpha Z(T_\alpha) = Z(\bigcup_\alpha T_\alpha)$.

Let $P \in \bigcap_\alpha Z(T_\alpha)$. Then $P \in Z(T_\alpha)$ for all α. Therefore $f(P) = 0$ for all α and for all $f \in T_\alpha$. Hence $f(P) = 0$ for all $f \in \bigcup_\alpha T_\alpha$. Therefore $P \in Z(\bigcup_\alpha T_\alpha)$. Hence $\bigcap_\alpha Z(T_\alpha) \subseteq Z(\bigcup_\alpha T_\alpha)$.

Let $P \in Z(\bigcup_\alpha T_\alpha)$. Then for all $f \in \bigcup_\alpha T_\alpha$, $f(P) = 0$. Hence for all α and all $f \in T_\alpha$, $f(P) = 0$. Therefore, for all α, $P \in Z(T_\alpha)$. Therefore $P \in \bigcap_\alpha Z(T_\alpha)$. Hence $\bigcap_\alpha Z(T_\alpha) \supseteq Z(\bigcup_\alpha T_\alpha)$.

Hence the intersection of any family of algebraic sets is a zero set and therefore algebraic. □

Lemma 2.27. *$\mathbf{A}^n$ and $\emptyset$ are algebraic sets.*

Proof. $\mathbf{A}^n$ is algebraic as it the zero set $Z(0)$. $\emptyset$ is algebraic as it can be expressed as the zero set of a constant function namely $Z(1)$. □

2.2 Zariski topology

Definition 2.28. We define the *Zariski topology* on $\mathbf{A}^n$ by taking the open subsets to be the complements of the algebraic sets.

Theorem 2.29. *The Zariski topology on $\mathbf{A}^n$ is a topology.*

Proof. We need to show the following conditions hold.

(i) The empty set $\emptyset$ and the whole space $\mathbf{A}^n$ are open.

(ii) Any finite intersection of open sets is open.

(iii) The union of open sets is open.

Proof of (i). By Lemma 2.27 the whole space $\mathbf{A}^n$ is an algebraic set. The complement of $\mathbf{A}^n$ is the $\emptyset$. Hence $\emptyset$ is open in the Zariski topology.
By Lemma 2.27, $\emptyset$ in $\mathbf{A}^n$ is an algebraic set. The complement of $\emptyset$ is the whole space $\mathbf{A}^n$. Hence $\mathbf{A}^n$ is open in the Zariski topology.

Proof of (ii). Let U_1 and U_2 be two open subsets. We will prove $U_1 \cap U_2$ is open. $(U_1 \cap U_2)^c = U_1^c \cup U_2^c$. U_1^c is an algebraic set and U_2^c is an algebraic set. Lemma 2.25 tells us $U_1^c \cup U_2^c$ is an algebraic set. Hence $(U_1 \cap U_2)^c$ is an algebraic set and therefore $U_1 \cap U_2$ is open. Let U_3 be another open subset. $(U_1 \cap U_2) \cap U_3$ is open. We can repeat this process for finitely many open sets and we see the intersection of finitely many open sets is open.

Proof of (iii). Let O be a set of open subsets. Let $D = \bigcup_{U \in O} U$. We would like to show that D is open in $\mathbf{A}^n$. This can be proved by showing D^c is an algebraic set.

$$\begin{aligned} D^c &= (\bigcup_{U \in O} U)^c \\ &= \bigcap_{U \in O} U^c \end{aligned}$$

Since U is an open subset, U^c is an algebraic set. By Lemma 2.26 the intersection of algebraic sets is an algebraic set. Hence D^c is an algebraic set and therefore D is an open subset.

Hence the Zariski topology is a topology. □

Lemma 2.30. *The algebraic sets in $\mathbf{A}^1$ are the finite subsets, $\emptyset$ and the whole space.*

Proof. Let Y be an algebraic set in $\mathbf{A}^1$. Then $Y = Z(T)$ for some $T \subseteq A$. We therefore have three cases to consider.

(i) $T = \emptyset$

(ii) $T = \{0\}$

(iii) T contains a non-zero polynomial

Note. If $T = \emptyset$ then $Z(\emptyset) = \mathbf{A}^1$.
If $T = \{0\}$ then $Z(0) = \mathbf{A}^1$.
If T contains a non-zero polynomial $f(x)$ we can write this as $f(x) = c(x - a_1)\cdots(x-a_n)$, $c, a_1, \ldots a_n \in K$. Then $Z(f) = \{a_1, \ldots a_n\}$ which is a finite set. Since $f \in T$ by Lemma 2.21 $Z(T) \subseteq Z(f)$. Therefore $Z(T)$ is a finite set.
Let $Y = \emptyset$ in $\mathbf{A}^1$. Then we have a subset $S \subset A$ namely $S = \{1\}$ such that $Y = Z(1)$. Let $W = \mathbf{A}^1$. Then we have a subset S_1 of A namely $S_1 = \{0\}$ such that $W = Z(0)$. Hence $\emptyset$ and $\mathbf{A}^1$ are algebraic sets. Let $X = \{a_1, \ldots, a_n\}$ be a finite subset in $\mathbf{A}^1$. Then we have a subset $T = \{f(x)\}$ where $f(x) = (x - a_1)\cdots(x - a_n)$, $a_1, \ldots a_n \in K$ such that $X = Z(f(x))$. □

Definition 2.31. Let X be a topological space. X is defined as Hausdorff if the following holds. For any two points P, Q, where $P \neq Q$, there exist open subsets U and V in X such that $P \in U$ and $Q \in V$ and $U \cap V = \emptyset$.

Theorem 2.32. *The Zariski Topology on* $\mathbf{A}^1$ *is not Hausdorff*

Proof. If we can show there are two points P and Q, $P \neq Q$ such that for all open sets U and V with $P \in U$ and $Q \in V$, $U \cap V$ is not empty then $\mathbb{A}^1$ is not Hausdorff. From the previous lemma we know the open sets in $\mathbb{A}^1$ are the empty set and the complements of finite subsets. Let U be the complement of the finite subset W_1 with $P \in U$. Let V be the complement of the finite subset W_2 with $Q \in V$. Then $U \cap V = W_1^c \cap W_2^c = (W_1 \cup W_2)^c$ where $W_1 \cup W_2$ is a finite subset. Hence we have $U \cap V$ is the complement of a finite subset. We identity $\mathbf{A}^1$ with K since elements in $\mathbf{A}^1$ are simply elements in K. Hence $\mathbf{A}^1$ is infinite. The complement of a finite set in $\mathbf{A}^1$ is therefore non-empty. Hence $U \cap V$ is non-empty. Therefore the Zariski topology on $\mathbf{A}^1$ is not Hausdorff.

□

Lemma 2.33. $\mathbf{A}^1$ *is irreducible in the Zariski topology.*

Proof. By definition 1.42, $\mathbf{A}^1$ is irreducible if it cannot be expressed as the union of two proper closed subsets of $\mathbf{A}^1$. Lemma 2.30 tells us the only algebraic sets of $\mathbf{A}^1$ are the finite subsets, $\emptyset$ and $\mathbf{A}^1$. Hence the proper closed subsets of $\mathbf{A}^1$ are the finite sets. $\mathbf{A}^1$ is infinite hence it cannot be expressed as the union of two proper closed subsets of $\mathbf{A}^1$. Therefore $\mathbf{A}^1$ is irreducible. □

Before we discuss affine varieties it is necessary to look at the relationship between subsets of $\mathbf{A}^n$ and ideals in A.

2.3 Ideals of A and subsets of $\mathbf{A}^n$

Definition 2.34. For any subset $Y \subseteq \mathbf{A}^n$, we define the *ideal* of Y in A as

$$I(Y) = \{f \in A | f(P) = 0 \text{ for all } P \in Y\}$$

With this definition and definitions 2.20 and 2.24 we obtain a map from subsets of A to algebraic sets and a map from subsets of $\mathbf{A}^n$ to ideals. The properties of these maps are given in Lemma 2.35 to Lemma 2.40. One other property of these maps was proved earlier in Lemma 2.21.

Lemma 2.35. *If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{A}^n$, then $I(Y_2) \subseteq I(Y_1)$.*

Proof. Let $f \in I(Y_2)$ then for all $P \in Y_2$, $f(P) = 0$. Since $Y_1 \subseteq Y_2$, let $P \in Y_1$, then $f(P) = 0$ hence $f \in I(Y_1)$. Therefore $I(Y_2) \subseteq I(Y_1)$. □

Lemma 2.36. *For any two subsets Y_1, Y_2 of $\mathbf{A}^n$, then $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*

Proof. $\supseteq$ Let $f \in I(Y_1) \cap I(Y_2)$ then $f \in I(Y_1)$ and $f \in I(Y_2)$. Hence $f(P) = 0$ for all $P \in Y_1$ and $f(Q) = 0$ for all $Q \in Y_2$. $I(Y_1)$ and $I(Y_2)$ are sets of polynomials who vanish for all points in Y_1 and Y_2. So if f vanishes for all points P and also for all points Q then f will vanish for all points in $Y_1 \cup Y_2$ Hence $f \in I(Y_1 \cup Y_2)$. Therefore $I(Y_1) \cap I(Y_2) \subseteq I(Y_1 \cup Y_2)$.
$\subseteq$. Let $g \in I(Y_1 \cup Y_2)$. Then g vanishes for all $P \in Y_1 \cup Y_2$. $Y_1 \subseteq Y_1 \cup Y_2$ so by Lemma 2.35 $I(Y_1 \cup Y_2) \subseteq I(Y_1)$ hence $g \in I(Y_1)$. $Y_2 \subseteq Y_1 \cup Y_2$ so by Lemma 2.35 $I(Y_1 \cup Y_2) \subseteq I(Y_2)$ hence $g \in I(Y_2)$. Therefore $g \in I(Y_1) \cap I(Y_2)$ and $I(Y_1 \cup Y_2) \subseteq I(Y_1) \cap I(Y_2)$.

Hence $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$ as required. □

Definition 2.37. The radical of an ideal $\mathfrak{a}$ is defined as,

$$\sqrt{\mathfrak{a}} = \{f \in A | f^r \in \mathfrak{a} \text{ for some } r > 0\} \qquad (2)$$

Definition 2.38. Let R be a ring and I an ideal in R. I is a *radical ideal* if $I = \sqrt{I}$.

Lemma 2.39. *For any ideal $\mathfrak{a} \subseteq A$, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Proof. $\supseteq$. Let $f \in \sqrt{\mathfrak{a}}$ then by definition $f^r \in \mathfrak{a}$. f^r will vanish for all points of $Z(\mathfrak{a})$ hence f will also vanish for all points of $Z(\mathfrak{a})$. Therefore $f \in I(Z(\mathfrak{a}))$ and $I(Z(\mathfrak{a})) \supseteq \sqrt{\mathfrak{a}}$
$\subseteq$ Hilberts Nullstellensatz [SLA, Page 380]tells us that for an ideal $\mathfrak{a}$ in A and $f \in A$ a polynomial that vanishes for all points of $Z(\mathfrak{a})$ that $f^r \in \mathfrak{a}$ for some integer $r > 0$. This means for $f \in I(Z(\mathfrak{a}))$, f is also in the radical. Hence $\sqrt{\mathfrak{a}} \supseteq I(Z(\mathfrak{a}))$.

Therefore $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ □

Lemma 2.40. *For any subset $Y \subseteq \mathbf{A}^n$, $Z(I(Y)) = \bar{Y}$ where $\bar{Y}$ is the closure of Y.*

Proof. $\supseteq$ $I(Y) = \{f \in A$ such that $f(P) = 0$ for all $P \in Y\}$. The zero set of $I(Y)$ is all points Q in $\mathbf{A}^n$ such that $f(Q) = 0$ for all $f \in I(Y)$. Hence $Y \subseteq Z(I(Y))$. Since $Z(I(Y))$ is a closed set containing Y and $\bar{Y}$ is the intersection of all closed sets that contain Y then $\bar{Y} \subseteq Z(I(Y))$.

$\subseteq$. Let W be any closed set containing Y. Then $\bar{Y} = \bigcap W$ for W closed and $Y \subseteq W$. We want to show $\bar{Y} \supseteq Z(I(Y))$. In other words $\bigcap W \supseteq Z(I(Y))$. That is $Z(I(Y))$ has to be in all W to be in the intersection. Since W is a closed set of $\mathbf{A}^n$ it is an algebraic set and hence $W = Z(T)$ for $T \subseteq A$. Since $W \supseteq Y$ we have $Y \subseteq Z(T)$. By Lemma 2.35 $I(Z(T)) \subseteq I(Y)$. Also $T \subseteq I(Z(T))$. This gives us $T \subseteq I(Z(T)) \subseteq I(Y)$. Hence $T \subseteq I(Y)$. By Lemma 2.21 $Z(I(Y) \subseteq Z(T)$. Therefore $Z(I(Y) \subseteq W$.

Hence $Z(I(Y)) = \bar{Y}$.

□

Definition 2.41. Let R be a commutative ring. $r \in R$ is a *nilpotent element* if there exists a positive integer n such that $r^n = 0_R$ where 0_R is the zero in R.

Definition 2.42. A ring R is said to be *reduced* if it contains no non-zero nilpotent elements

Theorem 2.43. *Let R be a ring and $I \subseteq R$ an ideal. I is radical if and only if R/I is reduced.*

Proof. Let I be a radical ideal. This means $\sqrt{I} = I$. Suppose R/I contains non-zero nilpotent elements. Let $r + I$ be a non-zero nilpotent element in R/I. By definition 2.41, $(r+I)^n = 0_{R/I}$ for n a positive integer. This gives $r^n+I = 0+I$. Hence $r^n \in I$. Therefore $r \in \sqrt{I} = I$. Hence $r \in I$. Therefore, $r + I = 0 + I$ which is a contradiction to the assumption that $r+I$ was non-zero. Hence R/I contains no non-zero nilpotent elements showing R/I is reduced.

Suppose R/I is reduced. Let $r \in \sqrt{I}$. Then $r^n \in I$ for some $n \in \mathbb{Z}^+$. Hence $r^n + I = 0_{R/I}$. Therefore $(r + I)^n = 0_{R/I}$. Given that R/I is reduced, $r + I$ is nilpotent. Therefore $r + I = 0 + I$. Hence $r \in I$. Therefore $\sqrt{I} = I$. Hence I is a radical ideal. □

If K is not algebraically closed, then these results do not hold. For example, if $K = \mathbb{R}$, then the curve $x^2 + y^2 + 1 = 0$ has no roots in $\mathbb{R}$. $x^2 \geq 0$, $y^2 \geq 0$ for $x, y \in \mathbb{R}$ hence $x^2 + y^2 + 1 > 0$. Therefore, it has no points in $\mathbf{A}^2$. So $Z(x^2 + y^2 + 1) = \emptyset$. Let $\mathfrak{a}$ be the ideal generated by f. We note here that $\mathfrak{a}$ does not contain the constant polynomial 1 hence $\mathfrak{a} \neq A$. By Lemma 2.22 $Z(f) = Z(\mathfrak{a})$. Given that $Z(x^2 + y^2 + 1) = \emptyset$ we would have $Z(\mathfrak{a}) = \emptyset$. Hence $I(Z((\mathfrak{a}))) = I(\emptyset) = A$. $A \neq \mathfrak{a}$. Therefore $A \neq \sqrt{\mathfrak{a}}$. Hence Lemma 2.39 would be false.

Definition 2.44. An irreducible closed subset $C \subseteq \mathbf{A}^n$ is said to be minimal if $C \neq \emptyset$ and there exist no irreducible closed subset $S \subset \mathbf{A}^n$ such that $\emptyset \subsetneq S \subsetneq C$.

Lemma 2.45. *Let K be an algebraically closed field. Every minimal irreducible closed subset of $\mathbf{A}^n$ is a point.*

Proof. Let C be a minimal irreducible closed subset of $\mathbf{A}^n$ which consists of more than one point say $C = \{(a_1, \ldots a_n), (b_1 \ldots b_n), \ldots\}$. If C is minimal it means there is no irreducible closed subset S such that $\emptyset \subsetneq S \subsetneq C$. We need to show that the sets containing one point are closed and algebraic. Let $S = \{(a_1 \ldots a_n)\}$ be a set containing one point. If there exists $T \subseteq A$ such that $S = Z(T)$ then S is algebraic. Taking $\mathbf{A}^2$ as an example we let $P = (a_1, a_2)$. $T = \{x_1 - a_1, x_2 - a_2\}$. Let $Q = (x, y)$ be an arbitrary point. Is Q in $Z(T)$? If it is then $f(Q) = 0$ for each $f \in T$. So we have $x - a_1 = 0$ which gives $x = a_1$. $y - a_2 = 0$ which gives $y = a_2$, hence Q is only in $Z(T)$ if $Q = P$. Hence any point in $\mathbf{A}^2$ is closed. In general case let $P = (a_1, \ldots, a_n)$, then $T = \{x_1 - a_1, \ldots x_n - a_n\}$. Hence we have found a subset T such that $S = Z(T)$. The smallest closed subsets of $\mathbf{A}^n$ are sets containing one point. If S is not irreducible then we could express it as the union of two proper subsets of S but the only proper subset of S is the empty set. Hence S is irreducible. Therefore we have found a non empty irreducible closed subset which is contained in but not equal to C hence C is not minimal. Therefore every minimal irreducible closed subset of $\mathbf{A}^n$ is a point.

□

Theorem 2.46. *Let $f \in K[x_1 \ldots x_n]$ and $I = (x_1 - a_1 \ldots x_n - a_n)$ for $a_1 \ldots a_n \in K$. Then $f \equiv C$ Mod (I) for some $C \in K$.*

Proof. For the constant polynomials $t \in K[x_1 \ldots x_n]$, we have $t \equiv t$ Mod (I). For non-constant polynomials we have $f_1 \equiv c_1$ Mod I, $f_2 \equiv c_2$ Mod I. In fact for any $f_i \in K[x_1 \ldots x_n]$, $f_i \equiv c_i$ Mod I. Through addition and multiplication, we obtain $f_1 + f_2 \equiv c_1 + c_2$ Mod I and $f_1 f_2 \equiv c_1 c_2$ Mod I.

We can deduce the following. For any polynomial $f \in K[x_1 \ldots x_n]$, $f \equiv C$ Mod (I) for some $C \in K$.

□

Theorem 2.46 tells us that all $f \in K[x_1 \ldots x_n]$ can be expressed as $f = g_1(x_1 - a_1) + \ldots g_n(x_n - a_n) + C$ with $C \in K$ a constant and $g_1 \ldots g_n \in K[x_1 \ldots x_n]$.

Definition 2.47. Let R be a ring and $I \subset R$ an ideal. I is a *maximal ideal* if $I \neq R$ and if for any ideal $J \supset I$ then either $J = R$ or $J = I$.

Lemma 2.48. *Ideals of the form $I = (x_1 - a_1, \ldots x_n - a_n)$ are maximal ideals of A.*

Proof. We have a ring homomorphism from $A \to K$ namely the evaluation map $EV_P : f \mapsto f(P)$ of a point $P = (a_1 \ldots a_n)$. The kernel of EV_P is the set of all $g \in A$ such that $EV_P(g) = 0$. This means $g(P) = 0$. The polynomials $x_1 - a_1, x_2 - a_2 \ldots x_n - a_n$ are in the kernel for this evaluation map. And we know the kernel of a ring homomorphism is an ideal. So is the ideal generated by these elements in the kernel? This ideal consists of all elements of the form

$g_1f_1+g_2f_2+\cdots+g_nf_n$ where $g_i \in A$ and f_i are the generators. When we input our point P each term will become 0 since $EV_P(g_if_i) = g_if_i(P) = g_i(P)0 = 0$. Hence $(x_1 - a_1, \dots x_n - a_n) \subseteq \text{Ker}(EV_P)$. Let $f \in \text{Ker}(EV_P)$. By Theorem 2.46 $f = g_1(x_1-a_1)+\dots g_n(x_n-a_n)+C$. Since $f \in \text{Ker}(EV_P)$ then $f(P) = 0$. $f(P) = g_1(P)(0) + \cdots + g_n(P)(0) + C$. This gives $g_1(P)(0) + \cdots + g_n(P)(0) + C = 0$ hence $C = 0$. Therefore the elements in $\text{Ker}(EV_P)$ are of the form $g_1(x_1-a_1)+\dots g_n(x_n-a_n)$ which are precisely the elements in the ideal $(x_1 - a_1, \dots x_n - a_n)$ hence $Ker(EV_P) \subseteq (x_1-a_1, \dots x_n-a_n)$. Therefore $\text{Ker}(EV_P) = (x_1 - a_1, \dots x_n - a_n)$. We have a ring homomorphism from $A \to K$ and the ideal $I = (x_1 - a_1, x_2 - a_2 \dots x_n - a_n)$ its kernel. For each coset M of I the image under the evaluation map is an element of K and $E\bar{V}_P : M \mapsto EV_P(M)$ is a ring isomorphism of A/I onto the image of EV_P. [SL, Theorem 3.1 page 93] . But the image of EV_P is all of K since EV_P is surjective. For any element $k \in K$ there exists $f \in A$ such that $f(P) = k$ namely $f = k$. Hence A/I is isomorphic to K. Since K is a field this means A/I is a field. Therefore I is a maximal ideal of A. [SL, Theorem 3.4 page 96].

□

Definition 2.49. Let C and D be sets of sets. A map $f : C \longrightarrow D$ is inclusion reversing if for every pair $a_1, a_2 \in C$ where $a_1 \subseteq a_2$ then $f(a_2) \subseteq f(a_1)$.

Lemma 2.50. *There is a one to one inclusion-reversing correspondence between algebraic sets in $\mathbf{A}^n$ and radical ideals in A given by*

$$I : Y \mapsto I(Y)$$

and

$$Z : \mathfrak{a} \mapsto Z(\mathfrak{a})$$

Proof. First we show $I \circ Z$ and $Z \circ I$ are the respective identity maps. Let $\mathfrak{a}$ be a radical ideal.

$$(I \circ Z)(\mathfrak{a}) = I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$$

by Lemma 2.39. Since $\mathfrak{a}$ is radical, $\sqrt{\mathfrak{a}} = \mathfrak{a}$.
Let $Y \subseteq \mathbf{A}^n$ be an algebraic set.

$$(Z \circ I)(Y) = Z(I(Y))$$

By Lemma 2.40 $Z(I(Y)) = \bar{Y}$. Since Y is algebraic and a closed set $\bar{Y} = Y$.

Next we show the inclusion reversing.

Let $Y_1 \subseteq Y_2$. Then I maps $Y_1 \mapsto I(Y_1)$ and I maps $Y_2 \mapsto I(Y_2)$. By Lemma 2.35 $I(Y_2) \subseteq I(Y_1)$.
Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$. Then Z maps $\mathfrak{a}_1 \mapsto Z(\mathfrak{a}_1)$ and Z maps $\mathfrak{a}_2 \mapsto Z(\mathfrak{a}_2)$. By Lemma 2.21 $Z(\mathfrak{a}_2) \subseteq Z(\mathfrak{a}_1)$.

This shows there is a one to one inclusion-reversing correspondence between algebraic sets in $\mathbf{A}^n$ and radical ideals in A. □

2.4 Affine Varieties

Recall an algebraic set $Y \subseteq \mathbf{A}^n$ is *irreducible* if $Y \neq \emptyset$ and Y cannot be expressed as the union of two proper algebraic subsets.

Definition 2.51. Let R be a ring and $I \subset R$ an ideal. I is a *prime ideal* if $I \neq R$ and if $a, b \in R$ such that $ab \in I$ then either $a \in I$ or $b \in I$. Prime ideals are radical ideals.

Theorem 2.52. *An algebraic set is irreducible if and only if its ideal is a prime ideal.*

Proof. Let Y be an irreducible algebraic set, and let $I(Y)$ be its ideal. If $I(Y) = A$, then $I(Y) = I(\emptyset)$. But $Y \neq \emptyset$. Hence $I(Y) \neq A$.

Let $fg \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. Hence $Y = Y \cap (Z(f) \cup Z(g)) = (Y \cap Z(f)) \cup (Y \cap Z(g))$ both being closed subsets of Y. Since Y is irreducible then either $Y = Y \cap Z(f)$ or $Y = Y \cap Z(g)$. If $Y = Y \cap Z(f)$ then $Y \subseteq Z(f)$. If $Y = Y \cap Z(g)$ then $Y \subseteq Z(g)$. So either $f \in I(Y)$ or $g \in I(Y)$. Hence $I(Y)$ is prime.

Let $I(Y)$ be a prime ideal. If $Y = \emptyset$ then $I(Y) = I(\emptyset) = A$. But $I(Y)$ is prime hence $I(Y) \neq A$. Therefore $Y \neq \emptyset$. Assume Y is not irreducible. Then $Y = Y_1 \cup Y_2$ for two proper closed subsets of Y. Then $I(Y) = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$. Since $Y_1 \subsetneq Y$ then $I(Y) \subsetneq I(Y_1)$ by Lemma 2.50. Let $f \in I(Y_1) \backslash I(Y)$. Since $Y_2 \subsetneq Y$ then $I(Y) \subsetneq I(Y_2)$. Let $g \in I(Y_2) \backslash I(Y)$. Hence $fg \notin I(Y)$ since $I(Y)$ is prime. $I(Y_1)$ contains f and also fg. $I(Y_2)$ contains g and also fg. Hence $fg \in (I(Y_1) \cap I(Y_2)) \backslash I(Y)$. From our assumption that Y is not irreducible we deduced $I(Y) = I(Y_1) \cap I(Y_2)$ hence we have a contradiction and therefore Y must be irreducible.
Hence an algebraic set is irreducible if and only if its ideal is a prime ideal.

□

Lemma 2.53. *Let R be a ring and I an ideal of R. I is prime if and only if R/I is an integral domain.*

Proof. Let R be a ring and I an ideal of R. Assume R/I is an integral domain. $I \neq R$. Otherwise we would have that $R/I = R/R$. R/R is the set of all equivalence classes $\{[r + R] : r \in R\}$. This set contains only one element. The zero element. Hence we have the zero ring. The zero ring is not an integral domain. Therefore $I \neq R$. Let $ab \in I$, $a, b \in R$. Hence $ab + I = 0 + I$, Therefore $(a + I)(b + I) = 0 + I$. Since R/I is an integral domain then either $a + I = 0 + I$ or $b + I = 0 + I$. Hence either $a \in I$ or $b \in I$. This shows I is a prime ideal.

Let R be a ring and I a prime ideal of R. If $R/I = \{0\}$ then $I = R$. But I is prime hence $I \neq R$. Therefore $R/I \neq \{0\}$. Suppose $a + I, b + I$ are non zero elements of R/I such that $(a + I)(b + I) = 0 + I$. Hence $ab + I = 0 + I$. Hence $ab \in I$. Since I is prime this means either $a \in I$ or $b \in I$. In otherwords $a + I = 0 + I$ or $b + I = 0 + I$. Therefore R/I is an integral domain.

□

Example 2.54. Taking the polynomial ring $A = K[x_1, \dots x_n]$ and the zero ideal in A. By Lemma 2.53 the 0 ideal is prime if $A/0$ is an integral domain. $A/0$ is the set of equivalence classes $\{f + 0 : f \in A\}$. This set is simply A itself which is an integral domain. Hence the 0 ideal is prime.

Theorem 2.55. *Given an irreducible polynomial $f \in A = K[x_1 \dots x_n]$, $Y = Z(f)$ is irreducible.*

Proof. Since A is a UFD, f generates a prime ideal $\mathfrak{a}$ in A We have $Y = Z(f)$. Then $I(Y) = I(Z(f))$. By Theorem 2.22 $Z(f) = Z(\mathfrak{a})$ hence $I(Y) = I(Z(f)) = I(Z(\mathfrak{a}))$. By Lemma 2.39 $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. This means $I(Z(\mathfrak{a}))$ is the radical of $\mathfrak{a}$. $\mathfrak{a}$ is a prime ideal, equal to its own radical. i.e $I(Z(\mathfrak{a})) = \mathfrak{a}$. Hence $I(Z(f)) = \mathfrak{a}$ and by Theorem 2.52 $Z(f)$ is irreducible.

□

Lemma 2.56. $\mathbf{A}^n$ *is irreducible.*

Proof. $\mathbf{A}^n$ is an algebraic set. Its ideal $I(\mathbf{A}^n) = (0)$ the zero ideal. The zero ideal is prime. Hence by Theorem 2.52 $\mathbf{A}^n$ is irreducible. □

Definition 2.57. These irreducible algebraic sets(with the subspace topology) are called *affine varieties.*

Definition 2.58. A non-empty open subset of an affine variety is called a *quasi-affine variety.*

Definition 2.59. If $Y \subseteq \mathbf{A}^n$ is an affine algebraic set, we define the *affine coordinate ring* $A(Y)$ of Y to be $A/I(Y)$ where $A = K[x_1, \dots, x_n]$.

Lemma 2.60. *If Y is an affine variety, then $A(Y)$ is an integral domain.*

Proof. Let Y be an affine variety. This means Y is an irreducible algebraic set. By Theorem 2.52 its ideal $I(Y)$ is a prime ideal. By Lemma 2.53 $I(Y)$ is prime iff $A/I(Y)$ is an integral domain. Hence $A(Y)$ is an integral domain. □

Definition 2.61. A K*-algebra* is a commutative ring B that has the structure of a K-vector space and the following holds. If $x, y \in B$ and $a \in K$ then $(ax)y = a(xy) = x(ay)$

Lemma 2.62. *Let R be a K-algebra and I an ideal of R. Then R/I is a K-algebra.*

Proof. R/I is a commutative ring. So we need to show it is a K-vector space. We define an association,

$$\begin{aligned} K \times R/I &\longrightarrow R/I \\ \phi :(k, r + I) &\mapsto kr + I \end{aligned}$$

We check ϕ is well defined. Let $(k, r_1 + I), (k, r_2 + I) \in K \times R/I$ such that $(k, r_1 + I) = (k, r_2 + I)$. This gives us $r_1 + I = r_2 + I$. Hence $r_1 - r_2 \in I$. $k(r_1 - r_2) = k \cdot 1(r_1 - r_2) \in I$ since $r_1 - r_2 \in I$. $k \cdot 1(r_1 - r_2) = k(r_1 - r_2) =$

$kr_1 - kr_2 \in I$. Hence $kr_1 - kr_2 + I = (kr_1 + I) - (kr_2 + I) = 0 + I$. Hence $kr_1 + I = kr_2 + I$. Therefore ϕ is well defined.

Let 1 be the unit element in K and $r + I \in R/I$

$$\begin{aligned} 1 \cdot (r + I) &= 1r + I \\ &= r + I \end{aligned}$$

Let $k \in K$ and $r_1 + I, r_2 + I \in R/I$.

$$\begin{aligned} k(r_1 + I + r_2 + I) &= k(r_1 + r_2 + I) \\ &= k(r_1 + r_2) + I \\ &= kr_1 + kr_2 + I \\ &= (kr_1 + I) + (kr_2 + I) \\ &= k(r_1 + I) + k(r_2 + I) \end{aligned}$$

Let $k_1, k_2 \in K$ and $r + I \in R/I$.

$$\begin{aligned} (k_1 + k_2)(r + I) &= (k_1 + k_2)(r + I) \\ &= k_1 r + k_2 r + I \\ &= (k_1 r + I) + (k_2 r + I) \\ &= k_1(r + I) + k_2(r + I) \end{aligned}$$

$$\begin{aligned} (k_1 k_2)(r + I) &= k_1 k_2 r + I \\ k_1((k_2)(r + I)) &= k_1(k_2 r + I) \\ &= k_1 k_2 r + I \end{aligned}$$

$$\begin{aligned} (k(r_1 + I))(r_2 + I) &= (kr_1 + I)(r_2 + I) \\ &= (kr_1 r_2 + I) = k(r_1 r_2 + I) \\ &= k((r_1 + I)(r_2 + I)) \\ &= k((r_2 + I)(r_1 + I)) \\ &= (kr_2 + I)(r_1 + I) \\ &= (r_1 + I)(kr_2 + I) \end{aligned}$$

□

Definition 2.63. Given two K-algebras B and C, a *K-algebra homomorphism* is a map $F : B \longrightarrow C$ such that for all $k \in K$ and $x, y \in B$

- $F(kx) = kF(x)$
- $F(x+y) = F(x) + F(y)$
- $F(xy) = F(x)F(y)$
- $F(1_B) = 1_C$

Lemma 2.64. *Let $f : B \to C$ and $g : C \to D$ be K-algebra homomorphisms. Then $g \circ f : B \to D$ is a K-algebra homomorphism.*

Proof. It is known that the composition of two ring homomorphisms is a ring homomorphism. So we only have to show that for all $k \in K$ and $b \in B$ $(g \circ f)(kb) = k((g \circ f)(b))$. Let $k \in K$ and $b \in B$.

$$\begin{aligned}(g \circ f)(kb) &= g(f(kb)) \\ &= g(kf(b)) \\ &= k(g(f(b))) \\ &= k((g \circ f)(b))\end{aligned}$$

Hence $g \circ f$ is a K-algebra homomorphism. □

Definition 2.65. B is a *finitely generated K-algebra* if there is a K-algebra homomorphism from $K[x_1, \ldots x_n]$ onto B.

Theorem 2.66. *$A(Y)$ is a finitely generated K-algebra.*

Proof. There is a map from $K[x_1, \ldots x_n]$ onto $A(Y)$ namely $F : f \mapsto f \text{ Mod } I(Y)$. We check this is a K-algebra homomorphism. Let $k \in K$ and $f, g \in K[x_1 \ldots x_n]$. Then $F(kf) = kf \text{ Mod } I(Y) = k(f \text{ Mod } I(Y)) = kF(f)$.

$F(f+g) = (f+g) \text{ Mod } I(Y) = f \text{ Mod } I(Y) + g \text{ Mod } I(Y) = F(f) + F(g)$.

$F(fg) = fg \text{ Mod } I(Y) =(f \text{ Mod } I(Y))(g \text{ Mod } I(Y)) = F(f)F(g)$.

□

Lemma 2.67. *Any finitely generated K-algebra B which is an integral domain is the affine coordinate ring of some affine variety*

Proof. Let B be a finitely generated K-algebra. Then there is a K-algebra homomorphism F from $A = K[x_1, \ldots x_n]$ onto B. $A/Ker(F)$ is isomorphic to B [SL, Theorem 3.1 page 93]. Ker(F) is an ideal $\mathfrak{a}$. We are given that B is an integral domain hence $\mathfrak{a}$ is a prime ideal in A. Since $\mathfrak{a}$ is prime it is radical. Lemma 2.50 tells us there is a correspondence between radical ideals and algebraic sets. Hence $a = I(Y)$ for some algebraic set Y. Since $\mathfrak{a}$ is prime by Theorem 2.52 we know Y is irreducible and hence it is an affine variety. Therefore $B \cong A/I(Y)$. □

Lemma 2.68. *$f(x, y) = y - x^2$ is irreducible.*

Proof. Assume $y - x^2$ is reducible. $y - x^2$ is a polynomial of total degree 2. By assumption of reducibility, it can be expressed as the product of two linear factors. $y - x^2 = (ax + by + c)(dx + ey + g)$ for $a, b, c, d, e, g \in K$. Hence we would obtain,

$$\begin{aligned} y - x^2 &= (ax + by + c)(dx + ey + g) \\ &= adx^2 + aexy + agx + bdyx + bey^2 + bgy + cdx + cey + cg \\ &= adx^2 + bey^2 + (bd + ae)xy + (ag + cd)x + (bg + ce)y + cg \end{aligned}$$

Hence we would have the following equations.

$$\begin{aligned} ad &= -1 \\ be &= 0 \\ bd + ae &= 0 \\ ag + cd &= 0 \\ bg + ce &= 1 \\ cg &= 0 \end{aligned}$$

From $be = 0$ either $b = 0$ or $e = 0$.

Case 1: If $b = 0$, then we would obtain the following.

$$\begin{aligned} ad &= -1 \\ ae &= 0 \\ ag + cd &= 0 \\ ce &= 1 \\ cg &= 0 \end{aligned}$$

From $ae = 0$ either $a = 0$ or $e = 0$. We see from $ce = 1$ that $e \neq 0$. We also see from $ad = -1$ that $a \neq 0$. Hence $ae = 0$ is not true.

Case 2: If $e = 0$ then we would obtain the following equations.

$$\begin{aligned} ad &= -1 \\ bd &= 0 \\ ag + cd &= 0 \\ bg &= 1 \\ cg &= 0 \end{aligned}$$

From $bd = 0$ either $b = 0$ or $d = 0$. We see from $bg = 1$ that $b \neq 0$. We also see from $ad = -1$ that $d \neq 0$. Hence $bd = 0$ is not true.

Hence $y - x^2$ is irreducible. □

Lemma 2.69. *Let* $Y = Z(y - x^2)$. *Then* $A(Y) \cong K[t]$ *where* $K[t]$ *is the polynomial ring in one variable.*

Proof. $A(Y) = A/I(Y) = K[x, y]/I(Y)$. First we determine what $I(Y)$ is. From definition

$$I(Y) = \{f \in A | f(P) = 0 \text{ for all } P \in Y\}$$

Y is the zero set of $f = y - x^2$ we are looking for the $I(Z(f))$. Since $y - x^2$ is irreducible in $K[x, y]$ then it generates a prime ideal $(y - x^2)$. Every prime ideal is equal to its own radical. By Lemma 2.39 we have $I(Y) = I(Z(f)) = (y - x^2)$. So we have $A(Y) = K[x, y]/(y - x^2)$ and we want to show it is isomorphic to the polynomial ring in one variable over K. We can construct a homomorphism

$$K[x, y] \longrightarrow K[t]$$
$$\phi : f(x, y) \mapsto f(t, t^2)$$

but we would like a ring isomorphism from $\frac{K[x,y]}{(y-x^2)} \to K[t]$. This can be achieved if we can find a ring homomorphism

$$\bar{\phi} : \frac{K[x, y]}{(y - x^2)} \longrightarrow K[t]$$

and a ring homomorphism

$$\varphi : K[t] \longrightarrow \frac{K[x, y]}{(y - x^2)}$$

such that $\varphi \circ \bar{\phi}$ and $\bar{\phi} \circ \varphi$ are the respective identity mappings. Let $\bar{\phi}$ be the map

$$\bar{\phi} : f(x, y) + (y - x^2) \mapsto f(t, t^2)$$

We check that $\bar{\phi}$ is well defined. Let $f(x, y) + (y - x^2)$ and $g(x, y) + (y - x^2) \in K[x, y]/(y - x^2)$ and assume $f(x, y) + (y - x^2) = g(x, y) + (y - x^2)$ then we have $f(x, y) - g(x, y) \in (y - x^2)$ so $f(x, y) - g(x, y)$ is some polynomial multiple of

$y-x^2$. i.e $f(x,y)-g(x,y)=h(x,y)(y-x^2)$ for some $h\in K[x,y]$.

$$\begin{aligned}
\phi\big(f(x,y)-g(x,y)\big) &= \phi\Big(h(x,y)(y-x^2)\Big)\\
\phi\big(f(x,y)\big)-\phi\big(g(x,y)\big) &= \phi(h(x,y))\phi(y-x^2)\\
f(t,t^2)-g(t,t^2) &= h(t,t^2)(t^2-t^2)\\
f(t,t^2)-g(t,t^2) &= h(t,t^2)(0)\\
f(t,t^2)-g(t,t^2) &= 0\\
f(t,t^2) &= g(t,t^2)
\end{aligned}$$

Hence $\bar{\phi}$ is well defined.

Let $f(x,y)+(y-x^2)$ and $g(x,y)+(y-x^2)$ be elements of $K[x,y]/(y-x^2)$.

$$\begin{aligned}
\bar{\phi}(f(x,y)+(y-x^2)+g(x,y)+(y-x^2)) &= \bar{\phi}(f(x,y)+g(x,y)+(y-x^2))\\
&= \bar{\phi}((f+g)(x,y)+(y-x^2))\\
&= (f+g)(t,t^2)\\
&= f(t,t^2)+g(t,t^2)\\
&= \bar{\phi}(f(x,y)+(y-x^2))+\bar{\phi}(g(x,y)+(y-x^2))
\end{aligned}$$

$$\begin{aligned}
\bar{\phi}(f(x,y)+(y-x^2))(g(x,y)+(y-x^2)) &= \bar{\phi}(f(x,y)g(x,y)+(y-x^2))\\
&= \bar{\phi}(fg(x,y)+(y-x^2))\\
&= fg(t,t^2)\\
&= f(t,t^2)g(t,t^2)\\
&= \bar{\phi}(f(x,y)+(y-x^2))\bar{\phi}(g(x,y)+(y-x^2))
\end{aligned}$$

The multiplicative unit element in $K[x,y]/(y-x^2)$ is $1+(y-x^2)$. $\bar{\phi}(1+(y-x^2))=1$ and we see that the unit element from $K[x,y]/(y-x^2)$ maps to the unit element in $K[t]$.

Hence $\bar{\phi}$ is a ring homomorphism.

Let φ be the map
$$\varphi : f(t)\mapsto f(x)+(y-x^2)$$

Let $f(t), g(t) \in K[t]$.

$$\begin{aligned}
\varphi(f(t)+g(t)) &= \varphi((f+g)(t)) \\
&= (f+g)(x) + (y-x^2) \\
&= f(x) + g(x) + (y-x^2) \\
&= f(x) + (y-x^2) + g(x) + (y-x^2) \\
&= \varphi(f(t)) + \varphi(g(t))
\end{aligned}$$

$$\begin{aligned}
\varphi(f(t)g(t)) &= \varphi\big((fg)(t)\big) \\
&= (fg)(x) + (y-x^2) \\
&= f(x)g(x) + (y-x^2) \\
&= (f(x) + (y-x^2))(g(x) + (y-x^2)) \\
&= \varphi(f(t))\varphi(g(t))
\end{aligned}$$

The unit element in $K[t]$ is the polynomial 1. $\varphi(1) = 1 + (y - x^2)$. The unit element in $K[t]$ maps to the unit element in $K[x,y]/(y-x^2)$.

Hence φ is a ring homomorphism.

We have found two ring homomorphisms and we need to check that the condition is satisfied. Let $f(x,y) + (y-x^2) \in K[x,y]/(y-x^2)$ then

$$\begin{aligned}
\varphi \circ \bar{\phi}(f(x) + (y-x^2)) &= \varphi(f(t,t^2)) \\
&= f(x,x^2) + (y-x^2) \\
&= f(x,y) + (y-x^2)
\end{aligned}$$

$f(x,x^2) + (y-x^2) = f(x,y) + (y-x^2)$ since they are the same coset.

Proof. These cosets are equal since

$$
\begin{aligned}
y &\equiv x^2 \bmod (y - x^2) \\
y^j &\equiv (x^2)^j \bmod (y - x^2) \\
x^i y^j &\equiv x^i (x^2)^j \bmod (y - x^2) \\
a_{ij} x^i y^j &\equiv a_{ij} x^i (x^2)^j \bmod (y - x^2) \\
\sum_{i=0,j=0}^{n} a_{ij} x^i y^j &\equiv \sum_{i=0,j=0}^{n} a_{ij} x^i (x^2)^j \bmod (y - x^2)
\end{aligned}
$$

Therefore $f(x, x^2) + (y - x^2) = f(x, y) + (y - x^2)$. □

Hence $\varphi \circ \bar{\phi}$ is indeed the identity mapping in $K[x, y]/(y - x^2)$.
Let $f(t) \in K[t]$

$$
\begin{aligned}
\bar{\phi} \circ \varphi(f(t)) &= \bar{\phi}(f(x) + (y - x^2)) \\
&= f(t)
\end{aligned}
$$

$\bar{\phi} \circ \varphi$ is the identity mapping in $K[t]$.
Hence $\bar{\phi}$ is a ring isomorphism and $A(Y) \cong K[t]$ □

Lemma 2.70. *$f(x, y) = xy - 1$ is irreducible.*

Proof. Assume $xy - 1$ is reducible. $xy - 1$ is a polynomial of total degree 2. By assumption of being reducible it can be expressed as the product of two linear factors. $xy - 1 = (ax + by + c)(dx + ey + g)$ for $a, b, c, d, e, g \in K$. Hence we would obtain,

$$
\begin{aligned}
xy - 1 &= (ax + by + c)(dx + ey + g) \\
&= adx^2 + aexy + agx + bdyx + bey^2 + bgy + cdx + cey + cg \\
&= adx^2 + bey^2 + (bd + ae)xy + (ag + cd)x + (bg + ce)y + cg
\end{aligned}
$$

Hence we would have the following equations.

$$
\begin{aligned}
ad &= 0 \\
be &= 0 \\
bd + ae &= 1 \\
ag + cd &= 0 \\
bg + ce &= 0 \\
cg &= -1
\end{aligned}
$$

From $ad = 0$ either $a = 0$ or $d = 0$.

Case 1: If $a = 0$, then we would obtain the following.

$$\begin{aligned} be &= 0 \\ bd &= 1 \\ cd &= 0 \\ bg + ce &= 0 \\ cg &= -1 \end{aligned}$$

From $cd = 0$ either $c = 0$ or $d = 0$. We see from $bd = 1$ that $d \neq 0$. We also see from $gc = -1$ that $c \neq 0$. Hence $cd = 0$ is not true.

Case 2: If $d = 0$ then we would obtain the following equations.

$$\begin{aligned} be &= 0 \\ ae &= 1 \\ ag &= 0 \\ bg + ce &= 0 \\ cg &= -1 \end{aligned}$$

From $ag = 0$ either $a = 0$ or $g = 0$. We see from $cg = -1$ that $g \neq 0$. We also see from $ae = 1$ that $a \neq 0$. Hence $ag = 0$ is not true.

Hence $xy - 1$ is irreducible. □

Example 2.71. Let $f(x, y) = xy - 1 \in K[x, y]$ and $Y = Z(xy - 1)$. $A(Y) \ncong K[t]$.

Proof. In this example $A(Y) = K[x, y]/(xy - 1)$. To show $A(Y) \ncong K[t]$ we look at the units in $K[t]$ which we know to be the constant functions. Assume $f + (xy - 1), f \in K[x, y]$ is a unit in $K[x, y]/(xy - 1)$. Hence we have that for some $g + (xy - 1) \in K[x, y]/(xy - 1)$, $(f + (xy - 1))(g + (xy - 1)) = 1 + (xy - 1)$. Hence $fg + (xy - 1) = 1 + (xy - 1)$. This means $fg - 1 \in (xy - 1)$. We can simply choose $f = x, g = y$ since $xy - 1 \in (xy - 1)$. Hence $x + (xy - 1)$ and $y + (xy - 1)$ are units in $K[x, y]/(xy - 1)$. Assume $x + (xy - 1)$ is a constant function. Then $x + (xy - 1) = c + (xy - 1)$ for some $c \in K$. Therefore $x - c \in (xy - 1)$. But elements in $(xy - 1)$ are of the form $f \cdot (xy - 1), f \in K[x_1, \ldots x_n]$. This means the total degree of any element in $(xy - 1)$ is either 0 or greater than or equal to 2. $x - c$ has total degree 1 and hence cannot be in $(xy - 1)$. Therefore we have a contradiction and $x + (xy - 1)$ is a non-constant unit in $K[x, y]/xy - 1$. Therefore $A(Y) \ncong K[t]$. □

Lemma 2.72. *A K-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbf{A}^n$ for some n if and only if B is finitely generated K-algebra with no nilpotent elements*

Let B be a finitely generated K-algebra with no nilpotent elements. Then there is a K-algebra homomorphism F from $A = K[x_1, \dots x_n]$ onto B. [SL, Theorem 3.1] tell us $A/Ker(F)$ is isomorphic to B. Ker(F) is an ideal $\mathfrak{a}$.

Since B is reduced $K[x_1, \dots, x_n]/\mathfrak{a}$ is reduced. Hence $\mathfrak{a}$ is a radical ideal by Theorem 2.43. Lemma 2.50 tells us there is a correspondence between radical ideals and algebraic sets. Hence $\mathfrak{a} = I(Y)$ for some algebraic set Y. Hence $B \cong A(Y)$ for some algebraic set in $\mathbf{A}^n$.

Let $B \cong A(Y)$ for Y an algebraic set in $\mathbf{A}^n$ and B a K-algebra . We know from Theorem 2.66 that $A(Y)$ is a finitely generated K-algebra so we just need to show it is reduced. We know $A(Y) = K[x_1, \dots, x_n]/I(Y)$. There is a correspondence between radical ideals and algebraic sets hence Y corresponds to some radical ideal $\mathfrak{a}$. By Theorem 2.43 an $\mathfrak{a}$ is radical if and only if $K[x_1, \dots, x_n]/I(Y)$ is reduced. Therefore B is reduced where $B \cong A(Y)$ for Y an algebraic set in $\mathbf{A}^n$.

Lemma 2.73. $\mathbf{A}^n$ *is a noetherian topological space.*

Proof. Let $Y_1 \supseteq Y_2 \supseteq \dots$ be a descending chain of closed subsets. Then by Lemma 2.35 $I(Y_1) \subseteq I(Y_2) \subseteq \dots$ is an ascending chain of ideals in $A = K[x_1, \dots, x_n]$. We know A is a noetherian ring hence this chain of ideals will become stationary. For each i, $Y_i = Z(I(Y))$ hence $Y_1 \supseteq Y_2 \supseteq \dots$ will also become stationary. □

Lemma 2.74. *Every algebraic set in* $\mathbf{A}^n$ *can be expressed uniquely as a union of affine varieties, no one containing another.*

Proof. This is a direct consequence of Theorem 1.53. □

The dimension of an affine variety is its dimension as a topological space.

Example 2.75. The dimension of $\mathbf{A}^1$ is 1. From Lemma 2.30 the only irreducible closed subsets of $\mathbf{A}^1$ are single points and $\mathbf{A}^1$ itself.

Definition 2.76. In a ring R the *height* of a prime ideal $\mathfrak{p}$ is the supremum of all $n \in \mathbb{Z}$ such that there exists a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \mathfrak{p}_n = \mathfrak{p}$ of distinct prime ideals.

Lemma 2.77. *Let* $K[x_1, x_2]$ *be the polynomial ring in* 2 *variables. Let* $\mathfrak{p} = (x_1) \subset K[x_1, x_2]$ *be a prime ideal. Then the height of* $\mathfrak{p}$ *is* 1.

Proof. Suppose $\mathfrak{q} \subsetneq \mathfrak{p}$ is a prime ideal with $\mathfrak{q} \neq (0)$. Then there is $f \in \mathfrak{q}$ such that $f = x_1^n g$ for $g \notin x_1 K[x_1, x_2]$. Since $\mathfrak{q}$ is prime then either $x_1^n \in \mathfrak{q}$ or $g \in \mathfrak{q}$. Since $\mathfrak{q} \subsetneq \mathfrak{p}$ $g \in \mathfrak{q}$ is excluded. Hence x_1^n must be in $\mathfrak{q}$. Since $\mathfrak{q}$ is prime then $x_1 \in \mathfrak{q}$. This means $\mathfrak{p} \subseteq \mathfrak{q}$. Therefore $\mathfrak{q} = \mathfrak{p}$ which is a contradiction. Hence the only chain is $(0) \subsetneq (x_1) = \mathfrak{p}$. Hence the height of $\mathfrak{p}$ is 1. □

We define the dimension of $A = K[x_1, \dots, x_n]$ to be the supremum of the heights of all prime ideals.

Lemma 2.78. *The dimension of* $A = K[x_1, \dots, x_n]$ *is* n.

Proof. If K is a field and A a finitely generated K-algebra then, when A is an integral domain we have that

$$\dim A = \text{ height } \mathfrak{p} + \text{ dim } \frac{A}{\mathfrak{p}}$$

[MAT, Chapter 5, Section 14]. In this case $K[x_1, \ldots, x_n]$ is a finitely generated K-algebra and an integral domain. Hence using this Theorem and induction on n we can prove that dim $K[x_1, \ldots, x_n] = n$. From lemma 2.77 we have that the prime ideal $\mathfrak{p} = (x_1) \subset K[x_1, \ldots, x_n]$ has height 1. We will choose this as our prime ideal.

$P(n)$: Dim $K[x_1, \ldots x_n] = n$.

$P(1)$ Let $n = 1$. Then the formula tells us

$$\dim K[x_1] = \text{ height } (x_1) + \text{ dim } \frac{K[x_1]}{(x_1)}$$

But $\frac{K[x_1]}{x_1} \cong K$. And any field K has dimension 0. This is because the only ideals in K are the 0 ideal and K itself. Hence we have

$$\dim K[x_1] = 1 + 0 = 1$$

Assume $P(n-1)$ is true. This means dim $K[x_1, \ldots, x_{n-1}] = n - 1$.

Now we will prove $P(n)$ is true.

$$K[x_1, \ldots, x_n] = \text{ height } (x_1) + \text{ dim } \frac{K[x_1, \ldots x_n]}{(x_1)}$$

But $\frac{K[x_1, \ldots x_n]}{(x_1)} \cong K[x_2, \ldots x_n]$. $K[x_2, \ldots x_n]$ is simply (with renumbering) the polynomial ring in $n-1$ variables. By assumption the polynomial ring in $n-1$ variables has dimension $n-1$. Hence we obtain,

$$K[x_1, \ldots, x_n] = 1 + n - 1 = n$$

Hence we have dimension of $K[x_1, \ldots, x_n] = n$. □

Lemma 2.79. *If Y is an affine algebraic set then dim Y=dim $A(Y)$.*

Proof. Let Y be an affine algebraic set in $\mathbf{A}^n$. The dimension of Y is its dimension as a topological space. Let dim $Y = r$. Then we have a chain of irreducible closed subsets $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_r \subseteq Y$. By Theorem 2.52 the closed irreducible subsets of Y correspond to prime ideals of A containing $I(Y)$. Hence we have a corresponding chain of prime ideals containing $I(Y)$ of length r. These prime ideals in A that contain $I(Y)$ correspond to the prime ideals of $A(Y)$.
Hence dim Y=dim $A(Y)$. □

Lemma 2.80. *The dimension of $\mathbf{A}^n = n$*

Proof. $\mathbf{A}^n$ is an affine algebraic set. Hence by Lemma 2.79 the dimension of $\mathbf{A}^n$ is equal to the dimension of its affine coordinate ring.

$$\begin{aligned} dim\ \mathbf{A}^n &= dim\ \frac{K[x_1, \ldots, x_n]}{I(\mathbf{A}^n)} \\ &= dim\ \frac{K[x_1, \ldots, x_n]}{\emptyset} \\ &= dim\ K[x_1, \ldots, x_n] \\ &= n \end{aligned}$$

□

Theorem 2.81. *Let R be a ring and $I \subseteq R$ an ideal. Let $J \subseteq R$ be an ideal with $J \supseteq I$. Then J/I is an ideal in R/I.*

Proof. $J/I = \{j + I : j \in J\}$, $J/I \subseteq R/I$.

We need to prove

- If $j_1 + I, j_2 + I \in J/I$ then $(j_1 + I) + (j_2 + I) \in J/I$
- The zero element of R/I is in J/I
- If $r + I \in R/I$ $j + I \in J/I$ then $(r + I)(j + I) \in J/I$

Let $j_1 + I, j_2 + I \in J/I$. $(j_1 + I) + (j_2 + I) = j_1 + j_2 + I = (j_1 + j_2) + I$. $(j_1 + j_2) + I \in J/I$ since $j_1 + j_2 \in J$. Therefore $(j_1 + I) + (j_2 + I) \in J/I$

The zero element of R/I is $0 + I$ with $0 \in R$. Is this in J/I? J is an ideal of R therefore it contains the zero element of R. Hence J/I contains the zero of R/I.

Let $r + I \in R/I$, $j + I \in J/I$. $(r + I)(j + I) = rj + I$. Since $rj \in J$, $rj + I \in J/I$. □

From Theorem 2.81 we now have a map from the set of ideals $J \subseteq R$ with $J \supseteq I$ to the set of ideals in R/I.

Theorem 2.82. *Let R be a ring, J an ideal in R with $J \supseteq I$. The factor ring R/I Mod J/I is isomorphic to R/J.*

Proof. We will show there is a ring homomorphism from R/I onto R/J namely

$$\begin{aligned} F : R/I &\longrightarrow R/J \\ r + I &\mapsto r + J \end{aligned}$$

We check first that F is well defined. Let $r_1 + I, r_2 + I \in R/I$ such that $r_1 + I = r_2 + I$. This means $r_1 - r_2 \in I$ but $I \subseteq J$ therefore $r_1 - r_2 \in J$ hence $r_1 + J = r_2 + J$. Therefore F is well defined.

Let $r_1 + I, r_2 + I \in R/I$.

$$\begin{aligned} F(r_1 + I + r_2 + I) &= F((r_1 + r_2) + I) \\ &= (r_1 + r_2) + J \\ &= (r_1 + J) + (r_2 + J) \\ &= F(r_1 + I) + F(r_2 + I) \end{aligned}$$

$$\begin{aligned} F((r_1 + I)(r_2 + I)) &= F(r_1 r_2 + I) \\ &= r_1 r_2 + J \\ &= (r_1 + J)(r_2 + J) \\ &= F(r_1 + I)F(r_2 + I) \end{aligned}$$

$F(1 + I) = 1 + J$.

Hence F is a ring homomorphism.

The kernel of F is the set of all elements $r + I \in R/I$ such that $F(r+I) = 0_{R/J}$. $F(r + I) = r + J$. Hence we have $r + J = 0 + J$, $r \in J$ Therefore the ker$F = \{r + I : r \in J\}$ which is simply J/I. The image of $F = \{r + J : F(r+I) = r + J$. This means the set of elements $r + J$ such that $r \in R$, which is simply all of R/J. Hence R/I Mod J/I is isomorphic to R/J. [SL, Theorem 3.1]. □

Lemma 2.83. $S = \{$*set of ideals* $J \subseteq R$ *with* $J \supseteq I\}$,$T = \{$*set of ideals of* $R/I\}$ *The map*

$$\begin{aligned} F : S &\longrightarrow T \\ J &\mapsto J/I \end{aligned}$$

is bijective.

Proof. Let f be the ring homomorphism

$$\begin{aligned} R &\longrightarrow R/I \\ r &\mapsto r + I \end{aligned}$$

Let $M \subseteq R/I$ be an ideal and $m_1, m_2 \in f^{-1}(M)$. Then $f(m_1)$ and $f(m_2) \in M$. Since M is an ideal $f(m_1) + f(m_2) \in M$. f is a ring homorphism therefore $f(m_1 + m_2) = f(m_1) + f(m_2) \in M$. Hence $m_1 + m_2 \in f^{-1}(M)$.

Let $m \in f^{-1}(M)$ and $f(r) \in R/I$. Then $f(m) \in M$. Since M is an ideal of R/I then $f(r)f(m) \in M$. f is a ring homomorphism therefore $f(rm) = f(r)f(m) \in M$. Hence $rm \in f^{-1}(M)$. Then $f^{-1}(M) \subseteq R$ is an ideal that contains I since the inverse image $f^{-1}(M) = \{x \in R | f(x) \in M\}$. Letting $x \in I$ then $f(x) = x + I = 0 + I \in R/I$. Since M is an ideal in R/I, $0 + I \in M$. Hence $f(x) \in M$ therefore $x \in f^{-1}(M)$. Hence $I \subseteq f^{-1}(M)$ We can now define a map $G : T \to S$ which sends M to $f^{-1}(M)$. We will now show the one to one correspondence between S and T by showing $F \circ G$ and $G \circ F$ are the respective identity maps.

$(F \circ G)(M) = F(G(M)) = F(f^{-1}(M)) = f^{-1}(M)/I$.
But $f^{-1}(M)/I$ is simply M. We see this as follows.

$\subseteq$. Let $x \in f^{-1}(M)/I$. Then $x = r + I$ for some $r \in f^{-1}(M)$. Therefore $f(r) \in M$. But $f(r) = r + I = x$. Hence $x \in M$. Therefore $f^{-1}(M)/I \subseteq M$.

$\supseteq$. Let $m \in M$. Then $m = s + I$ for some $s \in R$. But $s + I = f(s)$. Therefore $m = f(s)$ Hence $s \in f^{-1}(M)$. Hence $s + I \in f^{-1}(M)/I$. Therefore $m \in f^{-1}(M)/I$. Hence $f^{-1}(M)/I \supseteq M$.

Hence $f^{-1}(M)/I = M$.

$(G \circ F)(J) = G(J/I) = f^{-1}(J/I)$.
$f^{-1}(J/I) = \{x \in R | f(x) \in J/I\}$ and $J/I = \{j + I | j \in J\}$. We want to check for which $x \in R$ is $x + I \in J/I$? Let $x + I = j + I$ for some $j \in J$. Then $x - j \in I$ for some $j \in J$. $I \subseteq J \subseteq R$ are ideals in R. So if $x - j = i$ for some $i \in I$ and $j \in J$ then $x = i + j \in J$. Therefore, the only elements $x \in R$ that satisfy this are elements in J. Hence $f^{-1}(J/I) = J$. □

If J is a prime ideal in R then R/J is an integral domain and hence R/I Mod J/I is an integral domain and J/I is a prime ideal in R/I. By Lemma 2.83 the map F takes prime ideals which contain I in R to prime ideals in R/I.

Lemma 2.84. *Let R be a noetherian ring and $I \subset R$ an ideal. R/I is noetherian.*

Proof. Let $J_1/I \subsetneq J_2/I \subsetneq \cdots \subsetneq J_r/I \ldots$ be a chain of ideals in R/I with $J_i \in R$. By Lemma 2.83 each $J_i/I \in R/I$ corresponds to an ideal $J_i \in R$. Hence we have a corresponding chain of ideals in R, namely $J_1 \supsetneq J_2 \supsetneq \cdots \supsetneq J_r \ldots$. This chain will become stationary in R as R is Noetherian. Hence the chain in R/I will become stationary. Hence R/I is Noetherian. □

Lemma 2.85. *Let $f : A \to B$ be a ring isomorphism with $I \subseteq A$ and $J \subseteq B$ ideals such that $f(I) = J$. Then f induces an isomorphism,*

$$\begin{aligned} f' : A/I &\to B/J \\ a + I &\mapsto f(a) + J \end{aligned}$$

Proof. Since f is an isomorphism there is an inverse map $g : B \to A$ such that $f \circ g = Id_B$ and $g \circ f = Id_A$. The map g is defined by $J \mapsto f^{-1}(J)$. We define $g' : B/J \to A/I$ by $b + J \mapsto f^{-1}(b) + I$.
Let $a + I \in A/I$.

$$\begin{aligned}(g' \circ f')(a+I) &= g'(f'(a+I)) \\ &= g'(f(a)+J) \\ &= f^{-1}(f(a))+I \\ &= a+I\end{aligned}$$

Let $b+J \in B/J$.

$$\begin{aligned}(f' \circ g')(b+J) &= f'(g'(b+J)) \\ &= f'(f^{-1}(b)+I) \\ &= f(f^{-1}(b))+J \\ &= b+J\end{aligned}$$

Hence f' is an isomorphism. $\square$

3 Projective Space

The following section looks at the projective space. To discuss the projective space we recall the following definition.

Definition 3.1. If M is a set with an equivalence relation and $a \in M$ we call $[a] = \{m \in M | m \sim a\}$ the equivalence class of a.

3.1 Projective n-space

Definition 3.2. The *projective n-space* over a field K denoted $\mathbf{P}^n$ is the set of equivalence classes of $(n+1)$ tuples $(a_0, \ldots, a_n)$ of elements of K, not all zero, under the equivalence relation $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for all $\lambda \in K, \lambda \neq 0$.

Lemma 3.3. $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ *if and only if there exists a non-zero* $\lambda \in K$ *such that* $a_i = \lambda b_i$ *defines an equivalence relation.*

Proof. We have $a_i = \lambda a_i$ with $\lambda = 1$. Therefore $(a_0, \ldots, a_n) \sim (a_0, \ldots, a_n)$ and we have this relation is reflexive.

Let $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$. Then there exists $\lambda \in K$ such that $a_i = \lambda b_i$. Multiplying both sides by λ^{-1} we obtain $\lambda^{-1} a_i = b_i$ which is the same as $b_i = \lambda^{-1} a_i$ hence we have $(b_0, \ldots, b_n) \sim (a_0, \ldots, a_n)$ and this relation is symmetric.

Let $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ and $(b_0, \ldots, b_n) \sim (c_0, \ldots, c_n)$. Then there exists some $\lambda \in k$ such that $a_i = \lambda b_i$ and λ_1 such that $b_i = \lambda_1 c_i$. From $a_i = \lambda b_i$ we have $\lambda^{-1} a_i = b_i$ but $b_i = \lambda_1 c_i$. Therefore $\lambda^{-1} a_i = \lambda_1 c_i$. This gives $a_i = \lambda \lambda_1 c_i$. Hence $(a_0, \ldots, a_n) \sim (c_0, \ldots, c_n)$. Therefore this relation is transitive.

Hence this is an equivalence relation as defined. □

An element of the projective n-space is called a point. If $P \in \mathbf{P}^n$ is a point, then any $(n+1)$ tuple $(a_0 \ldots a_n)$ in the equivalence class of P is called the set of homogeneous coordinates for P.

3.1.1 Graded Rings

Definition 3.4. Let I be an index set and for each $i \in I$ let G_i be a group. Then $G = \prod_{i \in I} G_i$ is the direct product with the group operation defined component wise. In other words, for two elements in G say $(g_i)_{i \in I}$ and $(h_i)_{i \in I}$ their product is $(g_i h_i)_{i \in I}$.

Definition 3.5. For a family of abelian groups $A_i, i \in I$, the direct sum $\bigoplus_{i \in I} A_i$ is a subgroup of the direct product. It consists of elements $(a_i)_i \in \prod_{i \in I} A_i$ such that a_i is the neutral element for all but finitely many i.

Definition 3.6. A *graded ring* is a ring R such that the additive group of the ring is a direct sum of the abelian groups R_d $d \geq 0$ with $R_d R_e \subseteq R_{d+e}$ where $d, e \in \mathbb{Z}$,$d, e \geq 0$ and the unit element of R in R_0. i.e R_0 is a subring of R.

The elements of R_d are called the homogeneous elements of degree d.

Definition 3.7. Let R be a graded ring and $\mathfrak{a}$ an ideal of R. We say $\mathfrak{a}$ is a *homogeneous ideal* if $\mathfrak{a} = \bigoplus_{d \geq 0} (\mathfrak{a} \cap R_d)$ (i.e the direct sum of its homogeneous parts.)

Lemma 3.8. *Let R be a graded ring and $\mathfrak{a} \subseteq R$ an ideal. $\mathfrak{a}$ is homogeneous if and only if it can be generated by homogeneous elements.*

Proof. Assume $\mathfrak{a}$ is generated by homogeneous elements x_i. That means $x_i \in \mathfrak{a} \cap R_{d_i}$. Then for any $a \in \mathfrak{a}$, a can be expressed as $a = r_1 x_{i_1} + r_2 x_{i_2} + \dots r_n x_{i_n}$ By the definition of ideal $r_j x_{i_j} \in \mathfrak{a}$. Since R is a graded ring and $r_j \in R$ then each r_j can be expressed as the sum of its homogeneous parts. So we have $a = \sum_j^n \sum_k^n r_{j_k} x_{i_j} = \sum r_{j_k} x_{i_j}$. By the definition of ideal $r_{j_k} x_{i_j} \in \mathfrak{a}$. Since r_{j_k} is homogeneous and x_{i_j} is homogeneous, so is $r_{j_k} x_{i_j}$. Hence $\mathfrak{a}$ is homogeneous.
Conversely, assume that $\mathfrak{a}$ is homogeneous. Let $T = \bigcup_i \mathfrak{a} \cap R_i$. (the set of homogeneous elements of $\mathfrak{a}$). Let $\mathfrak{b}$ be the ideal generated by T. We want to show $\mathfrak{a} = \mathfrak{b}$. $\mathfrak{b}$ is the set of elements $r_1 t_1 + r_2 t_2 + \cdots + r_n t_n$ with $r_i \in R$ and $t_i \in T$. Since $T \subseteq \mathfrak{a}$ each $t_i \in \mathfrak{a}$. Hence $\mathfrak{b} \subseteq \mathfrak{a}$.
Let $a \in \mathfrak{a}$. a can be expressed as $a = a_1 + a_2 + \cdots + a_n$ with each $a_i \in \mathfrak{a} \cap R_i$. Hence $a_i \in T$. We can also express a as $a = 1a_1 + 1a_2 + \cdots + 1a_n$ with $1 \in R$ the unit element and $a_i \in T$. Hence we see $a \in \mathfrak{b}$. Therefore $\mathfrak{a} \subseteq \mathfrak{b}$. Hence $\mathfrak{a} = \mathfrak{b}$. □

Lemma 3.9. *Let $\mathfrak{a}, \mathfrak{b}$ be homogeneous ideals. Then $\mathfrak{a} + \mathfrak{b}$ is a homogeneous ideal.*

Proof. Since $\mathfrak{a}, \mathfrak{b}$ are homogeneous they can be generated by a set of homogeneous elements. Let $T = \{x_i : i \in I\}$ be the set of homogeneous elements that generate $\mathfrak{a}$. Let $S = \{y_j : j \in J\}$ be the set of homogeneous elements that generate $\mathfrak{b}$. I and J are index sets. Then $T \cup S$ is the set $\{x_i, y_j | i \in I, j \in J\}$.The ideal $\mathfrak{a} + \mathfrak{b}$ is the set $\{a + b | a \in \mathfrak{a}, b \in \mathfrak{b}\}$. $T \cup S$ generates some homogeneous ideal $\mathfrak{c} \subseteq \mathfrak{a} + \mathfrak{b}$. $\mathfrak{c}$ is the ideal consisting of all elements of the form $c = \sum_i r_i x_i + \sum_j s_j y_j$ where $r_i, s_j \in R$. Let $d \in \mathfrak{a} + \mathfrak{b}$. Then $d = a + b$ with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Since $\mathfrak{a}$ is generated by T we have $a = r_1 x_1 + r_2 x_2 + \cdots + r_n x_n$. Similarly since $\mathfrak{b}$ is generated by S we have $b = s_1 y_1 + \cdots + s_m y_m$. Therefore we have $d = r_1 x_1 + r_2 x_2 + \cdots + r_n x_n + s_1 y_1 + \cdots + s_m y_m$. Hence $d = \sum r_i x_i + \sum s_j y_j$. Therefore $\mathfrak{a} + \mathfrak{b} \subseteq \mathfrak{c}$. Thus $\mathfrak{a} + \mathfrak{b} = \mathfrak{c}$. and $\mathfrak{a} + \mathfrak{b}$ is homogeneous. □

Lemma 3.10. *Let $\mathfrak{a}, \mathfrak{b}$ be homogeneous ideals. Then $\mathfrak{a}\mathfrak{b}$ is a homogeneous ideal.*

Proof. Let T be a set of homogeneous x that generate $\mathfrak{a}$ and S a set of homogeneous y that generate $\mathfrak{b}$. Let $TS = \{xy : x \in T, y \in S\}$. Then TS generates a homogeneous ideal $\mathfrak{c}$ consisting of all elements of the form $\sum rxy$ with $r \in R, x \in T, y \in S$. The ideal $\mathfrak{a}\mathfrak{b}$ is the set of all elements $a_1 b_1 + \dots a_n b_n$ with $a_i \in \mathfrak{a}, b_i \in \mathfrak{b}$. Hence $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$. Let $a_1 b_1 + \cdots + a_n b_n \in \mathfrak{a}\mathfrak{b}$ with $a_i \in \mathfrak{a}, b_i \in \mathfrak{b}$. So we can take arbitrary $a_i b_i$ and show it is in $\mathfrak{c}$. Since $\mathfrak{a}$ is generated by T we have

$a_i = r_1x_1 + r_2x_2 + \cdots + r_nx_n$. Similarly since $\mathfrak{b}$ is generated by S we have $b_i = s_1y_1 + \cdots + s_my_m$. Therefore we have

$$\begin{aligned}
a_ib_i &= (r_1x_1 + r_2x_2 + \cdots + r_nx_n)(s_1y_1 + \cdots + s_my_m) \\
&= r_1x_1(s_1y_1 + \cdots + s_my_m) + \cdots + r_nx_n(s_1y_1 + \cdots + s_my_m) \\
&= \sum_{i=1}^{n} r_ix_i(s_1y_1 + \cdots + s_my_m) \\
&= \sum_{i=1}^{n} r_ix_i \sum_{j=1}^{m} s_jy_j \\
&= \sum_{\substack{1\le i\le n \\ 1\le j\le m}} r_ix_is_jy_j \\
&= \sum_{\substack{1\le i\le n \\ 1\le j\le m}} r_is_jx_iy_j
\end{aligned}$$

with $r_is_j \in R, x_i \in T, y_j \in S$. Hence we have $a_ib_i \in \mathfrak{c}$. Since $\mathfrak{c}$ is an ideal, sums of arbitrary a_ib_i are in $\mathfrak{c}$. Hence $\mathfrak{ab} \subseteq \mathfrak{c}$. Therefore $\mathfrak{ab} = \mathfrak{c}$ and $\mathfrak{ab}$ is a homogeneous ideal. □

Lemma 3.11. *Let $\mathfrak{a}$ be a homogeneous ideal and $f \in \mathfrak{a}$. Then for $f = f_0 + f_1 + \cdots + f_n$, with $f_d \in R_d$ each $f_d \in \mathfrak{a}$.*

Proof. We are given $\mathfrak{a}$ a homogeneous ideal and $f \in \mathfrak{a}$. We are given $f = f_0 + f_1 + \cdots + f_r$ with each $f_d \in S_d$. By definition 3.7 $f = g_0 + g_1 + \cdots + g_m$ with $g_d \in \mathfrak{a} \cap S_d$. We obtain from this that $f_0 = g_0, f_1 = g_1, \ldots f_r = g_m$. $m = n$. Hence $f_d = g_d$. Therefore $f_d \in \mathfrak{a}$. □

Lemma 3.12. *Let $\mathfrak{a}, \mathfrak{b}$ be homogeneous ideals. Then $\mathfrak{a} \cap \mathfrak{b}$ is a homogeneous ideal.*

Proof. Let $f \in \mathfrak{a} \cap \mathfrak{b}$. Then $f \in \mathfrak{a}$ and $f \in \mathfrak{b}$. We can write $f = f_0 + f_1 + \cdots + f_r$. (sum of homogeneous parts). By Lemma 3.11, $f_d \in \mathfrak{a}$. By Lemma 3.11, $f_d \in \mathfrak{b}$. Hence each $f_d \in \mathfrak{a} \cap \mathfrak{b}$. Therefore $\mathfrak{a} \cap \mathfrak{b}$ is a homogeneous ideal. □

Lemma 3.13. *Let $\mathfrak{a}$ be an ideal. Assume that for all $f \in \mathfrak{a}$ with $f = f_0 + \cdots + f_t$ and $f_d \in S_d$ we have $f_d \in \mathfrak{a}$. Then $\mathfrak{a}$ is homogeneous.*

Proof. Let T be the set of homogeneous elements f_d of $\mathfrak{a}$. We claim T generates $\mathfrak{a}$. We want to show all $f \in \mathfrak{a}$ are of the form $r_1f_1 + r_2f_2 + \cdots + r_tf_t$. Let $f \in \mathfrak{a}$. Since S is graded we can decompose f into the sum of its homogeneous parts. $f = f_0 + f_1 + \cdots + f_s$ with $f_d \in S_d$. By assumption $f_d \in \mathfrak{a}$. Hence $f_d \in S_d \cap \mathfrak{a}$. $(S_d \cap \mathfrak{a}) \subseteq T$ hence $f_d \in T$. We can let $r_i = 1$ and write $f = 1f_1 + 1f_2 + \cdots + 1f_s$. Hence T generates $\mathfrak{a}$ therefore $\mathfrak{a}$ is homogeneous. □

Lemma 3.14. *Let R be a graded ring. Let $\mathfrak{a}$ be a homogeneous ideal. Then $\sqrt{\mathfrak{a}}$ is a homogeneous ideal.*

Proof. We prove this lemma using induction. Our statement $P(r)$ is as follows. $P(r)$: For every $f \in \sqrt{\mathfrak{a}}$ with deg $f \leq r$ all homogeneous parts of f are in $\sqrt{\mathfrak{a}}$. Let $f \in \sqrt{\mathfrak{a}}$. ie $f \in R$ such that $f^n \in \mathfrak{a}$. Since R is graded we can express $f = f_r + f_{r-1} + \cdots + f_0$ with $f_i \in R_i$. Want to show each $f_i \in \sqrt{\mathfrak{a}}$.

$f^n = (f_r + \cdots + f_0)^n \in \mathfrak{a}$
$f^n = (f_r)^n +$ lower terms. Then by Lemma 3.11 $(f_r)^n \in \mathfrak{a}$ hence $f_r \in \sqrt{\mathfrak{a}}$. Since $f, f_r \in \sqrt{\mathfrak{a}}$, $f - f_r \in \sqrt{\mathfrak{a}}$. $f - f_r = f_{r-1} + \cdots + f_0 \in \sqrt{\mathfrak{a}}$
We can assume for every $g \in \sqrt{\mathfrak{a}}$ with degree $\leq r-1$ all homogeneous parts of $g \in \sqrt{\mathfrak{a}}$. Applying the assumption to $g = f - f_r$ we get each $f_i \in \sqrt{\mathfrak{a}}$ for $i = 0 \ldots r-1$. We have shown $f_r \in \sqrt{\mathfrak{a}}$. Hence $P(r)$ is true. □

Definition 3.15. Let R be a graded ring. $R_+ = \bigoplus_{d>0} R_d$ is defined as the *irrelevant ideal*.

Lemma 3.16. *R_+ is a homogeneous ideal.*

Proof. Let 0_R be the neutral element of R. i.e the zero polynomial in R. 0_R is in each R_d for $d > 0$(each R_d is an additive abelian subgroup of R). Hence $0_R \in R_+$.
Let $f = f_1 + f_2 + \cdots + f_{n-1} + f_n$ and $g = g_1 + g_2 + \cdots + g_{m-1} + g_m$ be elements of R_+ with each $f_i \in R_i$ and $g_j \in R_j$. If $m < n$ $g = g_1 + g_2 + \cdots + g_{m-1} + g_m + g_{m+1} + \cdots + g_n$ with $g_{m+1} = g_{m+2} = \cdots = g_n = 0$
If $n < m$ then $f = f_1 + f_2 + \cdots + f_{n-1} + f_n + f_{n+1} + \cdots + f_m$ with $f_{n+1} = f_{n+2} = \cdots = f_m = 0$. This allows us to assume without loss of generality that $m = n$.

$$\begin{aligned} f + g &= f_1 + f_2 + \cdots + f_n + g_1 + \cdots + g_2 + \cdots + g_n \\ &= f_1 + g_1 + f_2 + g_2 + \cdots + f_n + g_n \end{aligned}$$

Hence $f + g \in R_+$. Let $r \in R$ and $f = f_1 + f_2 + \cdots + f_n \in R_+$. we want to show $rf \in R_+$. $rf = rf_1 + rf_2 + \cdots + rf_n$. Need to show each $rf_i \in R_+$. We can write r as the sum of its homogeneous parts. $r = r_0 + r_1 + \ldots r_n$. We now have $rf_i = (r_0 + r_1 + \ldots r_n)f_i = r_0 f_i + r_1 f_i + \ldots r_n f_i$. Since $1 \leq i \leq n$ the lowest degree is non zero. Hence $rf_i \in R_+$. □

Definition 3.17. Let R be a graded ring and $I \subseteq R$ an ideal. $s \in R/I$ is homogeneous of degree i iff $s = r + I$ for some $r \in R$ homogeneous of degree i.

Lemma 3.18. *Let R be a graded ring and $I \subseteq R$ a homogeneous ideal. Then R/I comes with a grading making it a graded ring.*

Proof. We have that $R = \bigoplus R_i$ and $I = \bigoplus R_i \cap I$. R_i is an abelian subgroup of R by definition of R being graded. $I \subseteq R$ is an ideal and hence an abelian subgroup of R. The intersection of two abelian subgroups of a ring is again an abelian subgroup. Hence $(R_i \cap I)$ is an abelian subgroup. $R_i/(R_i \cap I)$ is an abelian group. Let

$$\phi_i : \frac{R_i}{R_i \cap I} \to \frac{R}{I}$$
$$r + (R_i \cap I) \mapsto r + I$$

Let $r+(R_i \cap I), s+(R_i \cap I) \in R_i/(R_i \cap I)$ with $r, s \in R_i$. Assume $r+(R_i \cap I) = s + (R_i \cap I), r, s \in R_i$. Hence $r - s \in (R_i \cap I)$. In particular $r - s \in I$. Hence $r + I = s + I$. $\phi_i(r + (R_i \cap I)) = r + I$. $\phi_i(s + (R_i \cap I)) = s + I$. Therefore $\phi_i(r + (R_i \cap I)) = \phi_i(s + (R_i \cap I))$. Hence ϕ_i is well defined.

$$\begin{aligned}
\phi_i((r + (R_i \cap I)) + (s + (R_i \cap I))) &= \phi_i(r + s + (R_i \cap I)) \\
&= (r + s) + I \\
&= (r + I) + (s + I) \\
&= \phi_i(r + (R_i \cap I)) + \phi_i(s + (R_i \cap I))
\end{aligned}$$

Hence ϕ_i is a group homomorphism.

$$\begin{aligned}
Ker(\phi_i) &= \{r + (R_i \cap I) : \phi_i(r + (R_i \cap I)) = 0\} \\
&= \{r + (R_i \cap I) : r + I = 0 \text{ in } R/I\} \\
&= \{r + (R_i \cap I) : r + I = 0 + I\} \\
&= \{r + (R_i \cap I) : r \in I\} \\
&= \{r + (R_i \cap I) : r \in R_i \cap I\} \\
&= \{0 + (R_i \cap I)\} \\
&= 0 \text{ in } R_i/(R_i \cap I)
\end{aligned}$$

Hence ϕ_i is injective. ϕ_i is an isomorphism onto Im $(\phi_i) \subseteq R/I$. [SL, Theorem 3.1 page 93] . Hence each $\phi_i(R_i/(R_i \cap I))$ is a subgroup of R/I.

We have to show R/I is the internal direct sum of all $\phi_i(R_i/(R_i \cap I))$. Let $r + I \in R/I$. Hence $r = \sum_{i \geq 0} r_i$ for some $r_i \in R_i$ and $r_i = 0$ for all but finitely many $i \geq 0$.Therefore $r + I = (\sum_{i \geq 0} r_i) + I = \sum_{i \geq 0} (r_i + I)$. Let $r + I$ be expressed in 2 different ways. So $r + I = \sum_{i \geq 0} (r_i + I), r_i \in R_i$ with $r_i = 0$ for all but finitely many $i \geq 0$. and $r + I = \sum_{i \geq 0} (s_i + I), s_i \in R_i$ with $s_i = 0$ for all but finitely many $i \geq 0$. Hence we have:

$$\begin{aligned}
\sum_{i\geq 0}(r_i + I) &= \sum_{i\geq 0}(s_i + I)\\
\sum_{i\geq 0}(r_i + I) - \sum_{i\geq 0}(s_i + I) &= 0\\
\sum_{i\geq 0}\big((r_i + I) - (s_i + I)\big) &= 0\\
\sum_{i\geq 0}\big((r_i - s_i) + I\big) &= 0\\
\big(\sum_{i\geq 0}(r_i - s_i)\big) + I &= 0
\end{aligned}$$

Hence $\sum_{i\geq 0}(r_i - s_i) \in I$. Recall $I = \bigoplus R_i \cap I$. Hence $\sum_{i\geq 0} r_i - s_i = \sum_{i\geq 0} a_i$ with $a_i \in R_i \cap I$ and $a_i = 0$ for all but finitely many $i \geq 0$.
Therefore we can conclude $r_i, s_i, r_i - s_i, a_i \in R_i$. Hence $r_i - s_i = a_i$ because of uniqueness in R. Therefore $r_i - s_i \in R_i \cap I$. Hence $r_i + (R_i \cap I) = s_i + (R_i \cap I)$.

Let $r + I \in \phi_i(R_i/(R_i \cap I))$ and $s + I \in \phi_j(R_j/(R_j \cap I))$. we have $r \in R_i, s \in R_j$. $(r + I)(s + I) = rs + I$. Since R is graded $rs \in R_{i+j}$. Hence $rs + I \in \phi_{i+j}(R_{i+j}/(R_{i+j} \cap I))$. Therefore R/I is a graded ring. □

3.1.2 The polynomial ring in $n + 1$ variables

Let S be the polynomial ring $K[x_0 \dots x_n]$. We can make S a graded ring by taking S_d to be the set of all sums of monomials of total degree d. Monomials are polynomials of one term. eg $4xy$.

Definition 3.19. A *homogeneous polynomial* is a polynomial whose non-zero terms have the same total degree.
$f(x_0, x_1) = x_0^3 + 2x_0^2x_1 + 3x_0x_1^2 + y^3$ is a homogeneous polynomial of degree 3.
$f(x_0, x_1) = x_0^4 + 2x_0x_1^2 + 3x_0^2 + 2x_0x_1^3$ is not a homogeneous polynomial.

Let $S = K[x_0, x_1]$ and $f(x_0, x_1) = x_0^4 + 2x_0x_1^2 + 3x_0^2 + 2x_0x_1^3 + x_0x_1 \in S$. This is not a homogeneous polynomial but can be expressed as the sum of its homogeneous components. $f_2 = x_0x_1 + 3x_0^2$, $f_3 = 2x_0x_1^2$, $f_4 = x_0^4 + 2x_0x_1^3$.

3.2 Zero sets of homogeneous elements

Let f be a homogeneous polynomial. Then $f(\lambda a_0 \dots \lambda a_n) = \lambda^d f(a_0, \dots a_n)$. We can define a well defined function from $\mathbf{P}^n$ to $\{0, 1\}$ by $f(P) = 0$ if $f(a_0 \dots a_n) = 0$ and $f(P) = 1$ if $f(a_0 \dots a_n) \neq 0$.

Definition 3.20. Let T be any set of homogeneous elements of $S = K[x_0, \ldots, x_n]$. We define the zero set of T to be

$$Z(T) = \{P \in \mathbf{P}^n | f(P) = 0 \text{ for all } f \in T\}$$

Lemma 3.21. *If $T_1 \subseteq T_2$ are subsets of S^h where S^h denotes the set of homogeneous elements of S then $Z(T_2) \subseteq Z(T_1)$.*

Proof. Let $P \in Z(T_2)$ then for all $f \in T_2$, $f(P) = 0$. Since $T_1 \subseteq T_2$, let $f \in T_1$, then $f(P) = 0$ hence $P \in Z(T_1)$. Therefore $Z(T_2) \subseteq Z(T_1)$. □

Definition 3.22. If $\mathfrak{a}$ is a homogeneous ideal of R, then $Z(\mathfrak{a}) = Z(T)$ where T is the set of all homogeneous elements in $\mathfrak{a}$.

Lemma 3.23. *Let $T \subseteq S^h$. Let $\mathfrak{a}$ be the ideal generated by T. Then $Z(T) = Z(\mathfrak{a})$.*

Proof. Let $\mathfrak{a}^h$ be the set of all homogeneous elements of $\mathfrak{a}$. By definition 3.22 $Z(\mathfrak{a}^h) = Z(\mathfrak{a})$. $\mathfrak{a}^h \subseteq S^h$. We have $T \subseteq \mathfrak{a}^h$. Hence by Lemma 3.21 $Z(\mathfrak{a}^h) \subseteq Z(T)$. Hence $Z(\mathfrak{a}) \subseteq Z(T)$.
Let $P = [a_0 : a_1 : \cdots : a_n] \in \mathbf{P}^n$ such that $P \in Z(T)$. Let $g \in \mathfrak{a}^h$. This means $g = g_1 f_1 + g_2 f_2 + \cdots + g_n f_n$ with $g_i \in S$ homogeneous and $f_i \in T$. We choose a representative of $[a_0 : a_1 : \cdots : a_n]$ say $Q = (a_0, a_1, \ldots, a_n) \in \mathbf{A}^{n+1}$. Then $g(Q) = 0$. Hence $g(P) = 0$ in the homogeneous sense. Therefore $P \in Z(\mathfrak{a}^h)$. Therefore $Z(T) \subseteq Z(\mathfrak{a}^h)$. By definition 3.22 $Z(\mathfrak{a}) = Z(\mathfrak{a}^h)$. Hence $Z(T) \subseteq Z(\mathfrak{a})$. Therefore $Z(T) = Z(\mathfrak{a})$. □

Theorem 3.24. *Homogeneous Nullstellensatz: If $\mathfrak{a}$ is a homogeneous ideal and $f \in S$ a homogeneous polynomial with deg $f > 0$ such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ then $f^r \in \mathfrak{a}$ for some r.*

Proof. Let $\mathbf{A}^{n+1} \backslash \{0\} \to \mathbf{P}^n$ be the surjective map which maps $(a_0 \ldots a_n) \mapsto (a_0 \ldots a_n)$. Let $Q \in \mathbf{A}^{n+1} \backslash \{0\}$ with $Q \in Z(\mathfrak{a})$ and Q not the origin. Then $Q \in Z(T)$ where T is the set of homogeneous elements of $\mathfrak{a}$. Let $P \in \mathbf{P}^n$ be the image of Q. Then $P \in Z(T)$. By definition 3.22, $P \in Z(\mathfrak{a})$. $f(P) = 0$ by assumption hence $f(Q) = 0$. Therefore $f^r \in \mathfrak{a}$ for some $r > 0$ by the Nullstellensatz in affine space .

□

Definition 3.25. Let $Y \subseteq \mathbf{P}^n$. Y is an *algebraic set* if there exists a set T of homogeneous elements of S such that $Y = Z(T)$.

Lemma 3.26. *The union of two algebraic sets is algebraic.*

Proof. Let $Y_1 \subseteq \mathbf{P}^n$ and $Y_2 \subseteq \mathbf{P}^n$ be algebraic sets. Then $Y_1 = Z(T_1)$ for some $T_1 \subset S^h$ and $Y_2 = Z(T_2)$ for some $T_2 \subset S^h$. $Y_1 \cup Y_2 = Z(T_1) \cup Z(T_2)$. We will show $Z(T_1) \cup Z(T_2) = Z(T_1 T_2)$ where $T_1 T_2$ is the set of all products of homogeneous elements of T_1 by homogeneous elements of T_2. i.e products of homogeneous polynomials.

Let $P \in Y_1 \cup Y_2$. Then $P \in Y_1$ or $P \in Y_2$.

If $P \in Y_1$ then $f(P) = 0$ for all $f \in T_1$. Hence, we have $f(P) \cdot g(P) = 0 \cdot g(P)$. for all $g \in T_2$. $0 \cdot g(P) = 0$. Hence $P \in Z(T_1T_2)$.
If $P \in Y_2$ then $g(P) = 0$ for some $g \in T_2$. Hence we have $f(P) \cdot g(P) = f(P) \cdot 0$. for all $f \in T_1$. $f(P) \cdot 0 = 0$. Hence $P \in Z(T_1T_2)$.

Hence $Z(T_1 \cup T_2) \subseteq Z(T_1T_2)$.

Let $P \in Z(T_1T_2)$ such that $P \notin Y_1$. Then there exists $f \in T_1$, such that $f(P) \neq 0$. For all $g \in T_2$ we have $0 = (fg)(P) = f(P)g(P)$. Since $f(P) \neq 0$, we have $g(P) = 0$ for all $g \in T_2$. Hence $P \in Y_2$.
Let $P \in Z(T_1T_2)$ such that $P \notin Y_2$. Then there exists $g \in T_2$, such that $g(P) \neq 0$. For all $f \in T_1$ we have $0 = (fg)(P) = f(P)g(P)$. Since $g(P) \neq 0$, then $f(P) = 0$ for all $f \in T_1$. Hence $P \in Y_1$

Therefore $Z(T_1T_2) \subseteq Z(T_1 \cup T_2)$.

Therefore $Z(T_1 \cup T_2) = Z(T_1T_2)$ showing the union of two algebraic sets is algebraic. □

Lemma 3.27. *The intersection of any family of algebraic sets is algebraic.*

Proof. Let $\{Y_\alpha\}$ be a set of algebraic sets over an index α. Then each $Y_\alpha = Z(T_\alpha)$ for $T_\alpha \subseteq S^h$. $\bigcap_\alpha Y_\alpha = \bigcap_\alpha Z(T_\alpha)$. We will show $\bigcap_\alpha Z(T_\alpha) = Z(\bigcup_\alpha T_\alpha)$.

Let $P \in \bigcap_\alpha Z(T_\alpha)$. Then $P \in Z(T_\alpha)$ for all α. Therefore $f(P) = 0$ for all α and for all $f \in T_\alpha$. Hence $f(P) = 0$ for all $f \in \bigcup_\alpha T_\alpha$. Therefore $P \in Z(\bigcup_\alpha T_\alpha)$. Hence $\bigcap_\alpha Z(T_\alpha) \subseteq Z(\bigcup_\alpha T_\alpha)$.

Let $P \in Z(\bigcup_\alpha T_\alpha)$. Then for all $f \in \bigcup_\alpha T_\alpha$, $f(P) = 0$. Hence for all α and all $f \in T_\alpha$, $f(P) = 0$. Therefore, for all α, $P \in Z(T_\alpha)$. Therefore $P \in \bigcap_\alpha Z(T_\alpha)$. Hence $\bigcap_\alpha Z(T_\alpha) \supseteq Z(\bigcup_\alpha T_\alpha)$.

Hence the intersection of any family of algebraic sets is a zero set and therefore algebraic. □

Lemma 3.28. *$\emptyset$ and $\mathbf{P}^n$ are algebraic sets.*

Proof. If the empty set is algebraic then we can find some set T of homogeneous elements of $S = K[x_0, \cdots, x_n]$ such that $\emptyset = Z(T)$. Let $Y = \emptyset$ in $\mathbf{P}^n$. Then we have a subset $T \subset S$ namely $T = \{1\}$ such that $Y = Z(1)$. Let $W = \mathbf{P}^n$. Then we have a subset T_1 of S namely $T_1 = \{0\}$ such that $W = Z(0)$.

Hence $\emptyset$ and $\mathbf{P}^n$ are algebraic sets. □

Definition 3.29. We define the *Zariski topology* on $\mathbf{P}^n$ by taking the open subsets to be the complements of the algebraic sets.

Lemma 3.30. *The Zariski topology on $\mathbf{P}^n$ is a topology.*

Proof. We need to show the following conditions hold.

(i) The empty set $\emptyset$ and the whole space $\mathbf{P}^n$ are open.

(ii) Any finite intersection of open sets is open.

(iii) The union of open sets is open.

Proof of (i). By Lemma 3.28 the whole space $\mathbf{P}^n$ is an algebraic set. The complement of $\mathbf{P}^n$ is the $\emptyset$. Hence $\emptyset$ is open in the Zariski topology.
By Lemma 3.28 $\emptyset$ in $\mathbf{P}^n$ is an algebraic set. The complement of $\emptyset$ is the whole space $\mathbf{P}^n$. Hence $\mathbf{P}^n$ is open in the Zariski topology.

Proof of (ii). Let U_1 and U_2 be two open subsets. We will prove $U_1 \cap U_2$ is open. $(U_1 \cap U_2)^c = U_1^c \cup U_2^c$. U_1^c is an algebraic set and U_2^c is an algebraic set. Lemma 3.26 tells us $U_1^c \cup U_2^c$ is an algebraic set. Hence $(U_1 \cap U_2)^c$ is an algebraic set and therefore $U_1 \cap U_2$ is open. Let U_3 be another open subset. $(U_1 \cap U_2) \cap U_3$ is open. We can repeat this process for finitely many open sets and we see the intersection of finitely many open sets is open.

Proof of (iii). Let O be a set of open subsets. Let $D = \bigcup\limits_{U \in O} U$. We would like to show that D is open in $\mathbf{A}^n$. This can be proved by showing D^c is an algebraic set.

$$\begin{aligned} D^c &= (\bigcup_{U \in O} U)^c \\ &= \bigcap_{U \in O} U^c \end{aligned}$$

Since U is an open subset, U^c is an algebraic set. By Lemma 3.27 the intersection of algebraic sets is an algebraic set. Hence D^c is an algebraic set and therefore D is an open subset.

Hence the Zariski topology on $\mathbf{P}^n$ is a topology. □

Definition 3.31. For any subset $Y \subseteq \mathbf{P}^n$, we define the *homogeneous ideal* of Y in S denoted $I(Y)$ as $I(Y) = \bigoplus\limits_{d \geq 0} \{f \in S_d : f(P) = 0 \text{ for all } P \in Y\}$.

Lemma 3.32. *I(Y) is an ideal.*

Proof. $0_S \in I(Y)$ as each S_d contains 0_S.

Let $f, g \in I(Y)$. We can write $f = f_0 + f_1 + \cdots + f_n$ with $f_d \in S_d$ and $g = g_0 + g_1 + \cdots + g_n$ with $g_d \in S_d$. Since $f, g \in I(Y)$ this means $f_d(P) = 0$ and $g_d(P) = 0$ for all d and all $P \in Y.f - g = (f-g)_0 + (f-g)_1 + \cdots + (f-g)_n = (f_0 - g_0) + (f_1 - g_1) + \cdots (f_n - g_n)$ where each $f_d - g_d \in S_d$. Let $P = [a_0 : \cdots : a_n] \in Y$. We choose a representative $Q \in \mathbf{A}^{n+1}$ namely $Q = (a_0, \ldots, a_n)$. $f_d(P) = 0$ implies $f_d(Q) = 0$. $g_d(P) = 0$ implies $g_d(Q) = 0$. $(f_d - g_d)(Q) = f_d(Q) - g_d(Q) = 0 - 0 = 0$. Hence $(f_d - g_d)(P) = 0$ in the homogeneous sense. Hence $f - g \in I(Y)$.

Let $f \in S$ and $g \in I(Y)$. We can write $f = f_0 + f_1 + \cdots + f_n$ with $f_d \in S_d$ and $g = g_0 + g_1 + \cdots + g_n$ with $g_d \in S_d$. Since $g \in I(Y)$ then each $g_d(P) = 0$ for all d and all $P \in Y$. $fg = (fg)_0 + (fg)_1 + \cdots + (fg)_n$ with each $(fg)_n \in S_d$. Hence we have $(f_0 g_0) + (f_1 g_1) + \cdots (f_n g_n)$ where each $f_d g_d \in S_d$. Let $P = [a_0 : \cdots : a_n] \in Y$. We choose a representative $Q \in \mathbf{A}^{n+1}$ namely $Q = (a_0, \ldots, a_n)$. $g_d(P) = 0$ implies $g_d(Q) = 0$. $(f_d g_d)(Q) = f_d(Q) g_d(Q) = f_d(Q) \cdot 0 = 0$. Hence $(f_d g_d)(P) = 0$ in the homogeneous sense. Therefore $fg \in I(Y)$.

Hence $I(Y)$ is indeed an ideal. □

Lemma 3.33. *If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{P}^n$, then $I(Y_2) \subseteq I(Y_1)$.*

Proof. Let $f \in I(Y_2)$. $f = f_0 + \cdots + f_r$ with each $f_d \in S_d$. Since $f \in I(Y_2)$, $f_d(P) = 0$ for all $P \in Y_2$. Given that $Y_1 \subseteq Y_2$, we let $P \in Y_1$. Then $f_d(P) = 0$ for all $P \in Y_1$. Hence $f \in I(Y_1)$. Therefore $I(Y_2) \subseteq I(Y_1)$. □

Lemma 3.34. *For any two subsets Y_1, Y_2 of $\mathbf{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*

Proof. $\supseteq$ Let $f \in I(Y_1) \cap I(Y_2)$. $f = f_0 + \ldots f_r$ with each $f_d \in S_d$. $f \in I(Y_1)$ and $f \in I(Y_2)$. Hence $f_d(P) = 0$ for all $P \in Y_1$ and $f_d(Q) = 0$ for all $Q \in Y_2$. Hence $f \in I(Y_1 \cup Y_2)$. Therefore $I(Y_1) \cap I(Y_2) \subseteq I(Y_1 \cup Y_2)$.

$\subseteq$. Let $g \in I(Y_1 \cup Y_2)$. $Y_1 \subseteq Y_1 \cup Y_2$ so by Lemma 3.33 $I(Y_1 \cup Y_2) \subseteq I(Y_1)$ hence $g \in I(Y_1)$. $Y_2 \subseteq Y_1 \cup Y_2$ so by Lemma 3.33 $I(Y_1 \cup Y_2) \subseteq I(Y_2)$ hence $g \in I(Y_2)$. Therefore, $g \in I(Y_1) \cap I(Y_2)$. Therefore $I(Y_1 \cup Y_2) \subseteq I(Y_1) \cap I(Y_2)$.

Hence $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$ as required. □

Lemma 3.35. *If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$ then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$*

Proof. $\supseteq$. Let $f \in \sqrt{\mathfrak{a}}$. Then by definition $f^r \in \mathfrak{a}$. $f = f_0 + f_1 + \cdots + f_l$. We will prove using induction on l the following statement:

$P(l)$: For each $f \in \sqrt{\mathfrak{a}}$ with $f \in S_0 \oplus S_1 \oplus \cdots \oplus S_l$ we have $f \in I(Z(\mathfrak{a}))$.

$P(0)$: Let $l = 0$. Let $f \in \sqrt{\mathfrak{a}}$. Therefore $f^r \in a$ for some r. We can write $f = f_0$ with $f_0 \in S_0$. If $f_0 = 0$ we are done as $0 \in I(Z(\mathfrak{a})$.
Assume $f_0 \neq 0$. We have that $f_0^r \in \mathfrak{a}$. Hence $f_0^r(P) = 0$ for all $P \in Z(\mathfrak{a})$. Therefore $f_0(P) = 0$ for all $P \in Z(\mathfrak{a})$. Hence $f_0 = 0$. Therefore $f_0 \in I(Z(\mathfrak{a}))$. Hence $f \in I(Z(\mathfrak{a}))$.

Assume $P(l-1)$ is true. For each $g \in \sqrt{\mathfrak{a}}$ with $g \in S_0 \oplus S_1 \oplus \cdots \oplus S_{l-1}$ we have $g \in I(Z(\mathfrak{a}))$.

We will show $P(l)$ is true.
Let $f \in \sqrt{\mathfrak{a}}$. Therefore $f^r \in \mathfrak{a}$. $f = f_0 + f_1 + \cdots + f_l$ with each $f_d \in S_d$. Assume $f(l) \neq 0$. Put $h = f^r \in \mathfrak{a}$. $h = h_0 + h_1 + \cdots + h_{lr}$ with each $h_d \in S_d$. Since $\mathfrak{a}$ is homogeneous each $h_d \in \mathfrak{a}$. Therefore $h_d(P) = 0$ for all $P \in Z(\mathfrak{a})$. In particular $h_{lr} \in \mathfrak{a}$. Hence $h_{lr}(P) = 0$ for all $P \in Z(\mathfrak{a})$. $h_{lr} = f_l^r$, then $f_l^r \in \mathfrak{a}$. Hence $f_l \in \sqrt{\mathfrak{a}}$. We now have $h_{lr}(P) = f_l^r(P) = 0$ for all $P \in Z(\mathfrak{a})$. Hence $f_l(P) = 0$ for all $P \in Z(\mathfrak{a})$. Therefore $f_l \in I(Z\mathfrak{a}))$.
$f - f_l = f_0 + \cdots + f_{l-1}$. Hence $f - f_l = g$ from $P(l-1)$. Now $f \in \sqrt{\mathfrak{a}}, f_l \in \sqrt{\mathfrak{a}}$

hence $f - f_l \in \sqrt{\mathfrak{a}}$. Hence $g \in \sqrt{\mathfrak{a}}$. Therefore, by assumption, $f - f_l \in I(Z(\mathfrak{a}))$. So we have $f - f_l \in I(Z(\mathfrak{a}))$ and $f_l \in I(Z(\mathfrak{a}))$. Therefore $f - f_l + f_l \in I(Z(\mathfrak{a}))$. Therefore $f \in I(Z(\mathfrak{a}))$ as required.

$\subseteq$ Let $f \in S$ be a non-constant homogeneous polynomial. Hilberts homogeneous Nullstellensatz tells us that if f vanishes for all points of $Z(\mathfrak{a})$ then $f^r \in \mathfrak{a}$ for some integer $r > 0$. This means for non-constant homogeneous $f \in I(Z(\mathfrak{a}))$, f is also in the radical. In the case of non-homogeneous polynomials we can decompose them into the sum of their homogeneous parts. Let $f \in I(Z(\mathfrak{a})$. $f = f_0 + f_1 + \cdots + f_m$ with $f_d \in S_d$. By Lemma 3.11 each $f_d \in I(Z(\mathfrak{a}))$ for all d. Hence $f_d \in \sqrt{\mathfrak{a}}$ for $d > 0$. Therefore $f \in \sqrt{\mathfrak{a}}$. If $f_0 = 0$ then $f_0 \in \sqrt{\mathfrak{a}}$. If $f_0 \neq 0$ then $Z(\mathfrak{a}) = \emptyset$ which is excluded. Hence $\sqrt{\mathfrak{a}} \supseteq I(Z(\mathfrak{a}))$.

Therefore $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ □

Lemma 3.36. *For any subset $Y \subseteq \mathbf{P}^n$, $Z(I(Y)) = \bar{Y}$.*

Proof. $\supseteq$ $I(Y) = \bigoplus_{d \geq 0} \{f \in S_d, f(P) = 0 \text{ for all } P \in Y\}$. The zero set of $I(Y)$ is all points Q in $\mathbf{P}^n$ such that $f(Q) = 0$ for all $f \in I(Y)$. Hence $Y \subseteq Z(I(Y))$. Since $Z(I(Y))$ is a closed set containing Y and $\bar{Y}$ is the intersection of all closed sets that contain Y then $\bar{Y} \subseteq Z(I(Y))$.
$\subseteq$. Let W be any closed set containing Y. Then $\bar{Y} = \bigcap W$ for W closed and $Y \subseteq W$. We want to show $\bar{Y} \supseteq Z(I(Y))$. In other words $\bigcap W \supseteq Z(I(Y))$. That is $Z(I(Y))$ has to be in all W to be in the intersection. Since W is a closed set of $\mathbf{P}^n$ it is an algebraic set and hence $W = Z(T)$ for $T^h \subseteq S$. By Lemma 3.23 $Z(T) = Z(\mathfrak{a})$ for the ideal $\mathfrak{a}$ generated by T. $Y \subseteq Z(\mathfrak{a})$ therefore by Lemma 3.33 $I(Z(\mathfrak{a})) \subseteq I(Y)$. Since $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$ then $\mathfrak{a} \subseteq I(Y)$. By Lemma 3.21 $Z(\mathfrak{a}) \supseteq Z(I(Y))$. Therefore $W \supseteq Z(I(Y))$.

Hence $Z(I(Y)) = \bar{Y}$.

□

Lemma 3.37. *If Y is an algebraic set in $\mathbf{P}^n$ then $I(Y)$ is a homogeneous radical ideal.*

Proof. Let Y be an algebraic set. By definition this means $Y = Z(T)$ for T a set of homogeneous elements. Let this T be the set of homogeneous elements of the homogeneous ideal $\mathfrak{a}$ it generates. Hence by definition 3.22 $Z(\mathfrak{a}) = Z(T)$. So $Y = Z(\mathfrak{a})$. Taking I of both sides we have $I(Y) = I(Z(\mathfrak{a}))$. By Lemma 3.35 $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for $Z(\mathfrak{a}) \neq \emptyset$. We also have $\sqrt{\mathfrak{a}} = \sqrt{\sqrt{\mathfrak{a}}}$. Hence $I(Z(\mathfrak{a}))$ is a homogeneous radical ideal. If $Z(\mathfrak{a}) = \emptyset$ then $I(Z(\mathfrak{a})) = I(\emptyset) = S = K[x_0, \ldots, x_n]$. □

Lemma 3.38. *If $\mathfrak{a}$ is a homogeneous ideal, $Z(\mathfrak{a})$ is an algebraic set.*

Proof. .Let $\mathfrak{a} \subseteq R$ be a homogeneous ideal. Then by definition 3.22 $Z(\mathfrak{a}) = Z(T^h)$ where T^h is the set of homogeneous elements of $\mathfrak{a}$. $T^h \subseteq S^h$. Hence $Z(\mathfrak{a})$ is an algebraic set. □

Lemma 3.39. *Let R be a graded ring. Let $\mathfrak{a}$ be a homogeneous radical ideal. If $Z(\mathfrak{a}) = \emptyset$ then $\mathfrak{a} = R$ or R_+.*

Proof. By definition $\mathfrak{a} = \bigoplus_{d\geq 0} \mathfrak{a}_d$ where $\mathfrak{a}_d = \mathfrak{a} \cap R_d$.
If $d = 0$ there are two possibilities. $\mathfrak{a}_0 = 0$ or $\mathfrak{a}_0 = R_0$.
For $d \geq 1$ we will show $\mathfrak{a}_d = R_d$. It is clear $\mathfrak{a}_d \subseteq R_d$. Let $f \in R_d$. By Theorem 3.24 $f^r \in \mathfrak{a}$ for some r. Since $\mathfrak{a}$ is radical, $f \in \mathfrak{a}$. Hence $f \in \mathfrak{a}_d$. Therefore $R_d \subseteq \mathfrak{a}_d$.
Hence $R_d = \mathfrak{a}_d$.
Therefore either $\mathfrak{a} = 0 + \bigoplus_{d\geq 1} \mathfrak{a}_d = R_+$ or $\mathfrak{a} = R_0 + \bigoplus_{d\geq 1} \mathfrak{a}_d = R$. □

Lemma 3.40. *There is a one to one inclusion-reversing correspondence between algebraic sets in $\mathbf{P}^n$ and homogeneous radical ideals in R not equal to R_+ given by*

$$I : Y \mapsto I(Y)$$

and

$$Z : \mathfrak{a} \mapsto Z(\mathfrak{a})$$

Proof. By Lemmas 3.11 and 3.14 we know the maps are inclusion reversing. All we need to show is the bijection. That is to show that $I \circ Z$ and $Z \circ I$ are the respective identity maps.
Let $\mathfrak{a}$ be a homogeneous radical ideal.

$$(I \circ Z)(\mathfrak{a}) = I(Z(\mathfrak{a}))$$

$I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ by Lemma 3.35 for $Z(\mathfrak{a}) \neq \emptyset$. $\sqrt{\mathfrak{a}} = \mathfrak{a}$. If $Z(\mathfrak{a}) = \emptyset$, by Lemma 3.39 either $\mathfrak{a} = R$ or $\mathfrak{a} = R_+$. $\mathfrak{a} = R_+$ is excluded. So we have R corresponding to $\emptyset$.
Let Y be an algebraic set.

$$(Z \circ I)(Y) = Z(I(Y))$$

$Z(I(Y)) = \bar{Y}$ by Lemma 3.36. Since Y is algebraic and a closed set $\bar{Y} = Y$.
□

Lemma 3.41. *Let $f \in S = K[x_0, \ldots x_n]$. Assume $(f) \subseteq S$ is a homogeneous ideal. Then f is homogeneous.*

Proof. $f = f_0 + f_1 + \cdots + f_r$ with $f_d \in S$ homogeneous of degree d $f_n \neq 0$. By Lemma 3.11 $f_r \in (f)$. Hence $f_r = gf$ for some $g \in S, g \neq 0$. Note both f_r and f are degree r. Hence g must be constant. Therefore $f = f_r/g$ homogeneous of degree r. □

Definition 3.42. An algebraic set $Y \subseteq \mathbf{P}^n$ is *irreducible* if it cannot be expressed as the union of two proper algebraic subsets.

Lemma 3.43. *Let $\mathfrak{a}$ be a homogeneous ideal. Suppose for any two homogeneous elements f, g, with $fg \in \mathfrak{a}$ either $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$. Then $\mathfrak{a}$ is a prime ideal.*

Proof. We prove this lemma by induction. Our statement $P(r)$ is as follows.
$P(r)$: For all elements $f, g \in R$ with $fg \in \mathfrak{a}$ and degree $f+$ degree $g \leq r$ either $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$. (or both).

Let $f, g \in R$ with $fg \in \mathfrak{a}$. Since R is graded we can decompose f and g into the sum of their homogeneous parts. Let $f = \sum_{i=0}^{t} f_i$ and $g = \sum_{j=0}^{m} g_j$ with $m+t \leq r$. Then $fg = \sum f_i \sum g_j = \sum f_i g_j \in \mathfrak{a}$. We take the highest term $f_t g_m$. By Lemma 3.11 $f_t g_m \in \mathfrak{a}$. By assumption either $f_t \in \mathfrak{a}$ or $g_m \in \mathfrak{a}$. Assume f_t is in $\mathfrak{a}$. Then $fg - f_t g \in \mathfrak{a}$. This gives $(f - f_t)g \in \mathfrak{a}$. The degree of $f - f_t$ is at most $t-1$ and g has degree m. Hence the degree of $(f - f_t)g$ is at most $t + m - 1$. $t + m - 1 \leq r - 1$. By the inductive hypothesis either $f - f_t \in \mathfrak{a}$ or $g \in \mathfrak{a}$. If $g \in \mathfrak{a}$ we are done. If $f - f_t \in \mathfrak{a}$ then $f \in \mathfrak{a}$.
Assume g_m is in $\mathfrak{a}$. Then $fg - fg_m \in \mathfrak{a}$. This gives $(g - g_m)f \in \mathfrak{a}$. The degree of $g - g_m$ is at most $m - 1$ and f has degree t. Hence the degree of $(g - g_m)f$ is at most $t + m - 1$. $t + m - 1 \leq r - 1$. By the inductive hypothesis either $g - g_m \in \mathfrak{a}$ or $f \in \mathfrak{a}$. If $f \in \mathfrak{a}$ we are done. If $g - g_m \in \mathfrak{a}$ then $g \in \mathfrak{a}$. Hence $\mathfrak{a}$ is a prime ideal.

□

Lemma 3.44. *Let Y_1, Y_2 be closed subsets of $\mathbf{P}^n$ such that $Y_1 \subsetneq Y_2$. Then $I(Y_2) \subsetneq I(Y_1)$.*

Proof. Let $I(Y_1) = I(Y_2)$. Then $Z(I(Y_1)) = Z(I(Y_2))$. By Lemma 3.36 $Z(I(Y_1)) = \bar{Y_1}$ and $Z(I(Y_2)) = \bar{Y_2}$. Hence $\bar{Y_1} = \bar{Y_2}$. Therefore $Y_1 = Y_2$. Hence for $Y_1 \subsetneq Y_2$, $I(Y_2) \subsetneq I(Y_1)$. □

Lemma 3.45. *An algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.*

Proof. Let Y be an irreducible algebraic set. $I(Y) = \bigoplus_{d \geq 0} \{f \in S_d : f(P) = 0 \text{ for all } P \in Y\}$

Assume $f, g \in R$ are homogeneous with $fg \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. Hence $Y = Y \cap (Z(f) \cup Z(g)) = (Y \cap Z(f)) \cup (Y \cap Z(g))$ both being closed subsets of Y. Since Y is irreducible then either $Y = Y \cap Z(f)$ or $Y = Y \cap Z(g)$. If $Y = Y \cap Z(f)$ then $Y \subseteq Z(f)$. If $Y = Y \cap Z(g)$ then $Y \subseteq Z(g)$. So either $f \in I(Y)$ or $g \in I(Y)$. By Lemma 3.43 $I(Y)$ is prime.

Let Y be an algebraic set and $I(Y)$ a prime ideal. $Y \neq \emptyset$. Otherwise we would have $I(\emptyset) = K[x_0, \ldots, x_n]$ which cannot happen since $I(Y)$ is prime by assumption. Assume Y is not irreducible. Then $Y = Y_1 \cup Y_2$ for two proper closed subsets of Y. Then $I(Y) = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$ by Lemma 3.34. Since $Y_1 \subsetneq Y$, Lemma 3.44 tells us $I(Y) \subsetneq I(Y_1)$. Let $f \in I(Y_1) \backslash I(Y)$. Since $Y_2 \subsetneq Y$, Lemma 3.44 tells us $I(Y) \subsetneq I(Y_2)$. Let $g \in I(Y_2) \backslash I(Y)$. Hence

$fg \notin I(Y)$ since $I(Y)$ is prime. $I(Y_1)$ contains f and also fg. $I(Y_2)$ contains g and also fg. Hence $fg \in (I(Y_1) \cap I(Y_2)) \backslash I(Y)$. From our assumption that Y is not irreducible we deduced $I(Y) = I(Y_1) \cap I(Y_2)$ hence we have a contradiction and therefore Y must be irreducible.

□

Lemma 3.46. $\mathbf{P}^n$ *is irreducible.*

Proof. By Lemma 3.28 we have that $\mathbf{P}^n$ is an algebraic set. From Lemma 3.45 we know any algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible iff $I(Y)$ is a prime ideal. Here we have $Y = \mathbf{P}^n$. This means $\mathbf{P}^n \subseteq \mathbf{P}^n$ is irreducible iff $I(\mathbf{P}^n)$ is a prime ideal. So we need to show $I(\mathbf{P}^n)$ is a prime ideal.

$$I(\mathbf{P}^n) = \bigoplus_{d \geq 0} \{f \in S_d | f(P) = 0 \text{ for all } P \in \mathbf{P}^n\}$$
$$I(\mathbf{P}^n) = (0)$$

So we have that $I(\mathbf{P}^n)$ is the zero ideal which is prime by Example 2.54 . Hence $\mathbf{P}^n$ is irreducible. □

Lemma 3.47. $\mathbf{P}^n$ *is a noetherian topologial space.*

Proof. Let $Y_1 \supseteq Y_2 \supseteq \ldots$ be a descending chain of closed subsets. Then by Lemma 3.33 $I(Y_1) \subseteq I(Y_2) \subseteq \ldots$ is an ascending chain of ideals in $S = K[x_0, \ldots, x_n]$. By Lemma 2.5, S is a Noetherian ring hence this chain of ideals will become stationary. Lemma 3.36 tells us $\bar{Y_i} = Z(I(Y_i))$. But here $Y_i = \bar{Y_i}$ since each Y_i is closed. Hence for each i, $Y_i = Z(I(Y_i))$ therefore $Y_1 \supseteq Y_2 \supseteq \ldots$ becomes stationary. □

Lemma 3.48. *Every algebraic set in* $\mathbf{P}^n$ *can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its irreducible components.*

Proof. By Theorem 1.53 we have every non-empty closed subset Y in a noetherian topological space X can be expressed as a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets Y_i. If $Y_i \not\supseteq Y_j$ for $i \neq j$ then Y_i are uniquely determined and are called the *irreducible components* of Y. Since $\mathbf{P}^n$ is a noetherian topological space and algebraic sets are closed subsets of $\mathbf{P}^n$ then it follows that every algebraic set in $\mathbf{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. □

Definition 3.49. A *projective algebraic variety* is an irreducible algebraic set in $\mathbf{P}^n$ with the induced topology. We will refer to them simply as projective varieties.

Definition 3.50. A non-empty open subset of a projective variety is a *quasi-projective variety.*

The dimension of a projective variety is its dimension as a topological space.

Lemma 3.51. *The dimension of* $\mathbf{P}^n = n$.

Proof. We can form a chain of closed irreducible sets as follows:

$$\mathbf{P}^n \supsetneq Z(x_n) \supsetneq \cdots \supsetneq Z(x_3, \dots x_n) \supsetneq Z(x_2, \dots x_n) \supsetneq Z(x_1, \dots x_n)$$

This chain has length n. Hence we have that the dimension $\mathbf{P}^n \geq n$.
Assume there is another chain

$$\mathbf{P}^n \supsetneq Z_r \supsetneq \cdots \supsetneq Z_2 \supsetneq Z_1 \supsetneq Z_0 \neq \emptyset$$

By Lemma 3.40 we have a correspondence between algebraic sets and ideals. Hence our chain corresponds to a chain in S namely

$$I(Z_r) \subsetneq \cdots \subsetneq I(Z_2) \subsetneq I(Z_1) \subsetneq I(Z_0) \subsetneq S_+$$

This chain has length $r+1$. By Lemma 2.78 the dimension of $S = K[x_0, \dots x_n] = n+1$. Hence $r+1 \leq n+1$. Hence $r \leq n$. This shows dimension $\mathbf{P}^n = n$. $\square$

Definition 3.52. If $Y \subseteq \mathbf{P}^n$ is an algebraic set we define the *homogeneous coordinate ring* of Y to be $S(Y) = S/I(Y)$ where S is the polynomial ring $K[x_0 \dots x_n]$.

If $f \in S$ is a linear homogeneous polynomial, then the zero set of f is called a hyperplane. We denote the zero set of x_i by H_i, for $i = 0, \dots, n$.

Lemma 3.53. *Let* U_i^n *be the open set* $\mathbf{P}^n - H_i$. *Let* $Y \subseteq U_i^n$ *be a closed subset and* $\bar{Y}$ *its closure in* $\mathbf{P}^n$ *then* $Y = \bar{Y} \cap U_i^n$.

Proof. By definition of closure $Y \subseteq \bar{Y}$. $Y \subseteq U_i^n$. Hence $Y \subseteq \bar{Y} \cap U_i$.
Since $\mathbf{P}^n$ is a topological space $Y = C \cap U_i^n$ for some closed subset C in $\mathbf{P}^n$. $Y \subseteq C$ and by definition of closure $\bar{Y} \subseteq C$. Hence we have $\bar{Y} \cap U_i^n \subseteq C \cap U_i^n = Y$. Therefore $\bar{Y} \cap U_i^n \subseteq Y$.

Hence $Y = \bar{Y} \cap U_i^n$. $\square$

Lemma 3.54. *Let* U_i^n *be the open set* $\mathbb{P}^n - H_i$. *Then* $\mathbb{P}^n$ *is covered by the open sets* U_i^n.

Proof. To prove this lemma we need to pick a point in $\mathbb{P}^n$ and prove it is in at least one of these U_i^n. Let $P = (a_0, \dots, a_n)$ and suppose $a_i \neq 0$. We want to show $P \in U_i^n$. But $U_i^n = \mathbb{P}^n - H_i$. So we can show $P \in \mathbb{P}^n - H_i$ or equivalently show that $P \notin H_i$. H_i is the zero set of x_i. By substitution of a_i into x_i we get a_i which is non-zero by assumption. Hence $P \notin H_i$. $\square$

We define a mapping

$$\begin{aligned} \phi_i^n : & U_i^n \to \mathbb{A}^n \\ & P \mapsto Q \end{aligned}$$

where $P = (a_0, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots a_n) \in U_i^n, a_i \neq 0$ and $Q = \phi_i^n(P) = (\frac{a_0}{a_i}, \ldots \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i} \ldots, \frac{a_n}{a_i})$ with a_i/a_i omitted.

Lemma 3.55. *ϕ_i^n is well defined*

Proof. Let $P = (a_0, \ldots a_n)$ and $P' = (b_0, \ldots, b_n)$. $P, P' \in U_i^n$ and $P = P'$. $\phi_i(P) = (a_0/a_i, \ldots a_n/a_i)$ and $\phi_i^n(P') = (b_0/b_i, \ldots b_n/b_i)$. Since $P = P'$ this means they are in the same equivalence class hence there exists $\lambda \neq 0$ such that $b_0 = \lambda a_0, b_1 = \lambda a_1, \ldots b_n = \lambda a_n$. Therefore $\phi_i^n(P') = (\lambda a_0/\lambda a_i, \ldots \lambda a_n/\lambda a_i) = (a_0/a_i, \ldots a_n/a_i)$. Hence $\phi_i^n(P) = \phi_i^n(P')$ and ϕ_i^n is well defined. □

Lemma 3.56. *ϕ_i^n is a homeomorphism of U_i^n with its induced topology to $\mathbf{A}^n$ with its Zariski Topology.*

Proof. Let $P = (a_0, \ldots, a_n)$ and $P' = (b_0, \ldots, b_n)$. ϕ_i^n is injective if for $\phi_i^n(P) = \phi_i^n(P')$ then $P = P'$. Let $\phi_i^n(P) = \phi_i^n(P')$. $\phi_i(P) = (a_0/a_i, \ldots a_n/a_i)$ and $\phi_i^n(P') = (b_0/b_i, \ldots, b_n/b_i)$. This means we need to find $\lambda \neq 0$ such that $b_k = \lambda a_k$. We try $\lambda = b_i/a_i$ which we know is non zero. We know since $\phi_i^n(P) = \phi_i^n(P')$ that $a_k/a_i = b_k/b_i$ hence we have $b_k = a_k b_i/a_i = a_k\lambda$ as required. Hence ϕ_i^n is injective.

Let $Q \in \mathbf{A}^n$. ϕ_i^n is surjective if for all $Q \in \mathbf{A}^n$ we can find a point $P \in U_i^n$ such that $\phi_i^n(P) = Q$. Let $Q = (d_1, \ldots d_i \ldots d_n)$. We guess $P = (d_1, \ldots d_i, 1, d_{i+1}, \ldots d_n)$ with 1 in the i^{th} position. Then $\phi_i^n(P) = (d_1/1, \ldots d_i/1, d_{i+1}/1, \ldots d_n/1) = Q$. Hence ϕ_i^n is surjective.

We are left to show the closed sets of U_i^n are identified with the closed sets of $\mathbf{A}^n$ by the map ϕ_i^n. We define a map $\alpha : S^h \to A$ from the set S^h of homogeneous elements of S to $A = K[y_1 \ldots y_n]$ by taking $f \in S^h$ we set $\alpha(f) = f(y_1, \ldots y_i, 1, y_{i+1}, \ldots y_n)$. We define a map $\beta : A \to S^h$ for $g \in A$ of degree e by setting $\beta(g) = x_i^e g(x_0/x_i, x_1/x_i, \ldots x_{i-1}/x_i, x_{i+1}/x_i \ldots\ldots x_n/x_i)$. $\beta(g)$ is a homogeneous polynomial of degree e.

Let $Y \subseteq U_i^n$ be a closed subset and $\bar{Y}$ its closure in $\mathbf{P}^n$. $\bar{Y}$ is closed and hence an algebraic set. Therefore $\bar{Y} = Z(T)$ for a set of homogeneous elements T. Let $T' = \alpha(T)$. We can show $\phi_i^n(Y) = Z(T')$ as follows.

$\subseteq$ Let $P = (a_0 \ldots a_n) \in Y$ and $Q = \phi_i^n(P) = (a_0/a_i, \ldots, a_n/a_i)$. Since $P \in Y, P \in \bar{Y}$. But $\bar{Y} = Z(T)$ hence $P \in Z(T)$. This means for any $f \in T, f(P) = 0$. Let $g = \alpha(f)$ for $f \in T$. Want to prove $g(Q) = 0$.
$g(Q) = f(a_0/a_i, \ldots a_{i-1}/a_i, 1, a_{i+1}/a_i, \ldots a_n/a_i)$. Since f is homogeneous of some degree d then $g(Q) = (1/a_i)^d f(a_0 \ldots a_n) = (1/a_i)^d(0) = 0$. Hence $g(Q) = 0$ for all $g \in T'$. So $Q \in Z(g)$ and hence $Q \in Z(T')$. Therefore $\phi_i^n(Y) \subseteq Z(T')$.
$\supseteq$ Let $Q = (b_1, \ldots, b_n) \in Z(T')$. This means $g(Q) = 0$ for $g \in T'$. Want to show $Q \in \phi_i^n(Y)$ or since $Q = \phi_i^n(P)$ for $P = (b_1, \ldots b_i, 1, b_{i+1}, \ldots, b_n)$ we can show $P \in Z(T)$ and hence $P \in \bar{Y} \supseteq Y$. We know $g(Q) = 0$ for $g \in T'$. This means $g = \alpha(f)$ for $f \in T$. So we have $f(P) = f(b_1, \ldots b_i, 1, b_{i+1}, \ldots, b_n) =$

$(\alpha(f))(Q) = g(Q) = 0$. Hence $f(P) = 0$, therefore $P \in Z(T)$ but $\bar{Y} = Z(T)$ hence $P \in \bar{Y}$. Since $Y = \bar{Y} \cap U$ and $P \in U$, $P \in Y$. Therefore $Z(T') \subseteq \phi_i^n(Y)$.

Hence $Z(T') = \phi_i^n(Y)$.

Let W be a closed set of $\mathbf{A}^n$. Then $W = Z(T')$ for some $T' \subseteq A$. We can show $(\phi_i^n)^{-1}(W) = Z(\beta(T')) \cap U$ as follows:

$\subseteq$ Let $Q = (b_1, \ldots b_n) \in W$. Hence $(\phi_i^n)^{-1}(Q) = P$ and is of the form $P = (b_1, \ldots b_i, 1, b_{i+1} \ldots, b_n)$. Since $Q \in W$, $Q \in Z(T')$ for all $g \in T'$. This means $g(Q) = 0$. Let $f = \beta(g)$. Want to show $f(P) = 0$. $f = \beta(g) = x_i^e g(x_0/x_i, \ldots x_{i-1}/x_i, x_{i+1}/x_i \ldots x_n/x_i)$. Substituting in P we get $1^e g(b_1/1, b_2/1 \ldots b_n/1) = g(Q) = 0$. Hence $P \in Z(\beta(g))$ for all $g \in T'$. $(\phi_i^n)^{-1}(Q) = P \in U$. Hence $(\phi_i^n)^{-1}(Q) \in Z(\beta(T')) \cap U$. Therefore $(\phi_i^n)^{-1}(W) \subseteq Z(\beta(T')) \cap U$.
$\supseteq$ Let $P \in Z(\beta(T')) \cap U$. Then $P \in Z(\beta(T'))$ and $P \in U$. Since $P \in U_i^n$, P is of the form $P = (b_1, \ldots, 1, \ldots, b_n)$. Hence for $P = (b_1, \ldots, 1 \ldots, b_n) \in Z(\beta(T'))$, $f(P) = 0$ for all $f \in \beta(T')$. Want to show $P \in (\phi_i^n)^{-1}(W)$. We define the map φ_i^n from $\mathbf{A}^n \to \mathbf{P}^n$ as $(b_1 \ldots b_n) \mapsto (b_1, \ldots, 1, \ldots, b_n)$.

$(\phi_i^n)^{-1}(W)$ is the set of all elements $(\phi_i^n)^{-1}(Q)$ for all $Q \in W$. So we want to show $P = (\phi_i^n)^{-1}(Q)$ for some $Q \in W$ namely $Q = (b_1, \ldots b_n)$. We have $W = Z(T')$. Since $f \in \beta(T')$, $f = \beta(g)$ for $g \in T'$. $f(x_0, \ldots x_n) = x_i^e g(x_0/x_i, \ldots x_n/x_i)$ hence

$0 = f(P) = f(b_1, \ldots, 1, \ldots, , b_n) = 1^e g(b_1/1, \ldots b_n/1) = g(b_1, \ldots b_n) = g(Q)$. This means $g(Q) = 0$ for all $g \in T'$ hence $Q \in Z(T')$. Therefore $Q \in W$ and $P \in (\phi_i^n)^{-1}(W)$. $Z(\beta(T')) \cap U \subseteq (\phi_i^n)^{-1}(W)$. Hence $Z(\beta(T')) \cap U_i^n = (\phi_i^n)^{-1}(W)$.

Therefore ϕ_i^n as defined is a homeomorphism. □

4 Localization of rings

Lemma 4.1. *Let R be a ring and S a subset such that $1 \in S$ and S is closed under multiplication. Let $a, b \in R$ and $s, t \in S$. Then $(a, s) \sim (b, t) \leftrightarrow (at - bs)u = 0$ for some $u \in S$ defines an equivalence relation on $R \times S$.*

Proof. Let $a \in R, s \in S$. Then $(a, s) \sim (a, s)$. Since $(as - as)u = 0(u) = 0$. Hence the relation is reflexive.
Let $a, b \in R, s, t \in S$ such that $(a, s) \sim (b, t)$. Then,

$$\begin{aligned}(at - bs)u &= 0\\ atu - bsu &= 0\\ atu &= bsu\\ 0 &= bsu - atu\\ 0 &= (bs - at)u\end{aligned}$$

Therefore $(b, t) \sim (a, s)$ and the relation is symmetric. Let $(a, s) \sim (b, t)$ and $(b, t) \sim (c, v)$. Then $(at - bs)u = 0$ and $(bv - ct)w = 0$ for some $u, w \in S$. This gives $atu - bsu = 0$ and $bvw - ctw = 0$.

$$\begin{aligned}atu - bsu &= 0 & bvw - ctw &= 0\\ (atu - bsu)vw &= 0 & (bvw - ctw)su &= 0\\ atuvw - bsuvw &= 0 & bvwsu - ctwsu &= 0\\ atuvw &= bsuvw & bvwsu &= ctwsu\end{aligned}$$

Hence $atuvw = ctwsu$. This results in $(av - cs)tuw = 0$. Since S is closed $tuw \in S$. Therefore $(a, s) \sim (c, v)$. Hence we have an equivalence relation as defined. □

Definition 4.2. Let R be a ring and S a subset such that $1 \in S$ and S is closed under multiplication. The *localization* of R by S forms a new ring $S^{-1}R$. The elements in this ring are equivalence classes of pairs $(a, s), a \in R, s \in S$ under the equivalence relation defined in Lemma 4.1. Let $\frac{a}{s}$ denote the equivalence class of (a, s).

We will show $S^{-1}R$ is indeed a ring.

Lemma 4.3. *Let R be a ring, $S \subset R$ multiplicatively closed, $a \in R$ and $b, s \in S$. Then $\frac{a}{b} = \frac{as}{bs}$ in $S^{-1}R$.*

Proof. Recall $a/b = c/d$ in $S^{-1}R$ iff $(ad - cb)r = 0$ for some $r \in S$. Let $a \in R$ and $b, s \in S$. Then

$$\begin{aligned}abs - bas &= 0\\ (abs - bas)1 &= 0 \text{ for } 1 \in S\end{aligned}$$

Hence $\frac{a}{b} = \frac{as}{bs}$ in $S^{-1}R$. □

Lemma 4.4. *Let R be a ring, $S \subset R$ multiplicatively closed, $a, b \in R$ and $s, t \in S$. If $at = bs \in R$, then $\frac{a}{s} = \frac{b}{t}$ in $S^{-1}R$.*

Proof. We are given $at = bs$ in R. Hence $at - bs = 0$. Therefore $(at - bs)1 = 0$ for $1 \in S$. Hence $\frac{a}{s} = \frac{b}{t}$ in $S^{-1}R$. □

$S^{-1}R$ is the set of equivalence classes. i.e $S^{-1}R = \{\frac{a}{s} | a \in R, s \in S\}$. We define addition and multiplication on $S^{-1}R$ as follows:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$
$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

Lemma 4.5. *Addition and multiplication on $S^{-1}R$ is well defined*

Proof. Let $(a, s) \sim (c, v)$ and $(b, t) \sim (d, w)$. Then $(av - cs)u = 0$ and $(bw - dt)y = 0$ for some $u, y \in S$. We have the following:

$$\begin{aligned} (av - cs)u &= 0 \qquad & (bw - dt)y &= 0 \\ avu - csu &= 0 \qquad & bwy - dty &= 0 \\ avutwy - csutwy &= 0 \qquad & bwysvu - dtysvu &= 0 \end{aligned}$$

Adding both equations we obtain

$$\begin{aligned} avutwy - csutwy + bwysvu - dtysvu &= 0 \\ (avtw - cstw + bwsv - dtsv)uy &= 0 \\ (avtw + bwsv - cstw - dtsv)uy &= 0 \\ [(at + bs)vw - (cw - dv)st]uy &= 0 \end{aligned}$$

Hence $(at + bs, st) \sim (cw + dv, vw)$. Therefore addition is well defined. We have

$$\begin{aligned} (av - cs)u &= 0 \qquad & (bw - dt)y &= 0 \\ avu - csu &= 0 \qquad & bwy - dty &= 0 \\ avubwy - csubwy &= 0 \qquad & cbwysu - cdtysu &= 0 \\ avubwy &= csubwy \qquad & cbwysu &= cdtysu \end{aligned}$$

Therefore $avubwy = cdtysu$. Hence $avubwy - cdtysu = 0$. $(abvw - cdst)uy = 0$. Since S is closed $uy \in S$. Therefore $(ab, st) \sim (cd, vw)$ and multiplication is well defined. □

Lemma 4.6. *Let R be a ring and S a subset such that $1 \in S$ and S is multiplicatively closed. $S^{-1}R$ is a ring.*

Proof. Let $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$.

$$\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$$

$at + bs \in R, st \in S$. Hence $S^{-1}R$ is closed under addition.

We guess $\frac{0_R}{1}$ as our additive neutral element. It is an element in $S^{-1}R$ as $0_R \in R$ and $1 \in S$. Now we check it satisfies as a neutral element. Let $\frac{a}{b} \in S^{-1}R$. Then,

$$\begin{aligned}\frac{a}{b} + \frac{0_R}{1} &= \frac{a + 0_R b}{b} \\ &= \frac{a}{b}\end{aligned}$$

$$\begin{aligned}\frac{0_R}{1} + \frac{a}{b} &= \frac{0_R b + a}{b} \\ &= \frac{a}{b}\end{aligned}$$

Hence we have an additive neutral element in $S^{-1}R$.

Let $a \in R$. R is a ring hence $-a \in R$. Therefore $\frac{-a}{s} \in S^{-1}R$.

$$\frac{a}{s} + \frac{-a}{s} = \frac{as - as}{s} = \frac{0}{s} = 0$$

Let $\frac{a}{s}, \frac{b}{t}, \frac{c}{u} \in S^{-1}R$.

$$\begin{aligned}(\frac{a}{s} + \frac{b}{t}) + \frac{c}{u} &= \frac{at+bs}{st} + \frac{c}{u} \\ &= \frac{atu + bsu + cst}{stu} \\ &= \frac{atu + (bsu + cst)}{stu} \\ &= \frac{atu}{stu} + \frac{bsu+cst}{stu} \\ &= \frac{a}{s} + (\frac{b}{t} + \frac{c}{u})\end{aligned}$$

Let $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$.

$$\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st} = \frac{bs+at}{st} = \frac{b}{t} + \frac{a}{s}$$

Therefore $S^{-1}R$ is a commutative additive group.

$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$. $ab \in R, st \in S$. Hence $S^{-1}R$ is closed under multiplication.
$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} = \frac{ba}{ts} = \frac{b}{t} \cdot \frac{a}{s}$. Hence $S^{-1}R$ is commutative with respect to multiplication.

$$\begin{aligned}
(\frac{a}{s} \cdot \frac{b}{t})\frac{c}{u} &= \frac{ab}{st} \cdot \frac{c}{u} \\
&= \frac{abc}{stu} \\
&= \frac{a(bc)}{s(tu)} \\
&= \frac{a}{s} \cdot (\frac{b}{t} \cdot \frac{c}{u})
\end{aligned}$$

Therefore multiplication is associative.
We guess $\frac{1_R}{1}$ as our unit element. It is an element in $S^{-1}R$ as $1_R \in R$ and $1 \in S$. Now we check it satisfies as a unit element. Let $\frac{a}{b} \in S^{-1}R$. Then,

$$\frac{a}{b} \cdot \frac{1_R}{1} = \frac{a}{b}$$

$$\frac{1_R}{1} \cdot \frac{a}{b} = \frac{a}{b}$$

Hence we have a unit element in $S^{-1}R$.
Let $\frac{a}{s}, \frac{b}{t}, \frac{c}{u} \in S^{-1}R$.

$$\begin{aligned}
(\frac{a}{s} + \frac{b}{t})\frac{c}{u} &= (\frac{at+bs}{st}) \cdot \frac{c}{u} \\
&= \frac{atc+bsc}{stu}
\end{aligned}$$

$$\begin{aligned}
\frac{a}{s} \cdot \frac{c}{u} + \frac{b}{t} \cdot \frac{c}{u} &= \frac{ac}{su} + \frac{bc}{tu} \\
&= \frac{actu+bcsu}{sutu} \\
&= \frac{u}{u} \cdot \frac{act+bcs}{stu} \\
&= \frac{act+bcs}{stu}
\end{aligned}$$

Hence $(\frac{a}{s}+\frac{b}{t})\frac{c}{u} = \frac{a}{s}\cdot\frac{c}{u}+\frac{b}{t}\cdot\frac{c}{u}$. Since $S^{-1}R$ is commutative $\frac{a}{s}(\frac{b}{t}+\frac{c}{u}) = \frac{a}{s}\cdot\frac{b}{t}+\frac{a}{s}\cdot\frac{c}{u}$. Therefore $S^{-1}R$ is a ring. □

Lemma 4.7. *Let R be a ring and S a subset such that $1 \in S$ and S is multiplicatively closed. Let $f : R \to S^{-1}R$ be defined by $a \mapsto a/1$. Then f is a ring homomorphism.*

Proof. Let $a, b \in R$. $f(a+b) = (a+b)/1 = a/1 + b/1 = f(a) + f(b)$
$f(ab) = (ab)/1 = a/1 \cdot b/1 = f(a)f(b)$
$f(1) = 1/1 = 1$ □

Theorem 4.8. *Let $g : R \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for each $s \in S$. Then $h : S^{-1}R \to B$ which takes $a/s \mapsto g(a)g(s)^{-1}$ is a well defined ring homomorphism.*

Proof. Let $(a, s) \sim (b, t), a, b \in R, s, t \in S$. Hence $(at - bs)u = 0$ for some $u \in S$. Putting $(at - bs)u = 0$ through g we have $g((at - bs)u) = g(0)$.

$$\begin{aligned} g((at - bs)u) &= g(0) \\ g(atu - bsu) &= 0 \\ g(atu) - g(bsu) &= 0 \\ g(a)g(t)g(u) - g(b)g(s)g(u) &= 0 \\ g(a)g(t)g(u)g(u)^{-1} - g(b)g(s)g(u)g(u)^{-1} &= 0, u \in S, \text{ so } g(u) \text{ is a unit} \\ g(a)g(t) - g(b)g(s) &= 0 \\ g(a)g(t)g(t)^{-1}g(s)^{-1} - g(b)g(s)g(t)^{-1}g(s)^{-1} &= 0 \\ g(a)g(s)^{-1} - g(b)g(t)^{-1} &= 0 \end{aligned}$$

Therefore $g(a)g(s)^{-1} = g(b)g(t)^{-1}$ and h is well defined.

$$\begin{aligned} h(a/s + b/t) &= h((at + bs)/st) \\ &= g(at + bs)g(st)^{-1} \\ &= g(at)g(st)^{-1} + g(bs)g(st)^{-1} \\ &= g(a)g(t)g(t)^{-1}g(s)^{-1} + g(b)g(s)g(t)^{-1}g(s)^{-1} \\ &= g(a)g(s)^{-1} + g(b)g(t)^{-1} \\ &= h(a/s) + h(b/t) \end{aligned}$$

$$\begin{aligned} h(a/s \cdot b/t) &= h(ab/st) \\ &= g(ab)g(st)^{-1} \\ &= g(a)g(b)g(t)^{-1}g(s)^{-1} \\ &= g(a)g(s)^{-1}g(b)g(t)^{-1} \\ &= h(a/s)h(b/t) \end{aligned}$$

$h(s/s) = g(s)g(s)^{-1} = 1$. Therefore h is a well defined ring homomorphism. $\square$

Lemma 4.9. *Let B be an integral domain. $S \subseteq B - \{0\}$ be multiplicatively closed. Then*

$$\begin{aligned} \phi : S^{-1}B &\to Q(B) \\ \frac{b}{s} \quad &\mapsto \frac{b}{s} \\ (b \in B, s \in S) \qquad &(b \in B, s \in B - \{0\}) \end{aligned}$$

where $Q(B)$ is the quotient field of B is a well defined injective ring homomorphism.

Proof. Let the quotient field of B be denoted by $T^{-1}B$ with $T = B - \{0\}$. Let $\frac{b}{s}, \frac{c}{u} \in S^{-1}B, b, c \in B, s, u \in S$ with $\frac{b}{s} = \frac{c}{u}$. This means $x(bu - cs) = 0$ in B for some $x \in S$. Since $S \subseteq B - \{0\}$, then $x \in B - \{0\}$. Hence $x(bu - cs) = 0$ in B for some $x \in B - \{0\}$. Hence $\frac{b}{s} = \frac{c}{u}$ in $T^{-1}B$. Therefore $\phi(\frac{b}{s}) = \phi(\frac{c}{u})$. Hence ϕ is well defined.
Let $\phi(\frac{b}{s}) = \phi(\frac{c}{u})$ in $T^{-1}B$. This means $y(bu - cs) = 0$ in B for some $y \in T$. Since B is an integral domain then either $y = 0$ or $bu - cs = 0$. But $y \in T$ therefore $y \neq 0$. Hence $bu - cs = 0$, thus $\frac{b}{s} = \frac{c}{u}$ (in $S^{-1}B$). Therefore ϕ is injective.
Let $\frac{a}{s}, \frac{c}{u} \in S^{-1}B, a, c \in B, s, u \in S$.

$$\begin{aligned} \phi(\frac{a}{s} + \frac{c}{u}) &= \phi(\frac{au + cs}{su}) \\ &= \frac{au + cs}{su} \\ &= \frac{au}{su} + \frac{cs}{su} \\ &= \frac{a}{s} + \frac{c}{u} \\ &= \phi(\frac{a}{s}) + \phi(\frac{c}{u}) \end{aligned}$$

$$\begin{aligned}\phi(\frac{a}{s}\cdot\frac{c}{u}) = \frac{ac}{su} \\ = \frac{a}{s}\cdot\frac{c}{u} \\ = \phi(\frac{a}{s})\phi(\frac{c}{u})\end{aligned}$$

$\phi(\frac{1}{1}) = \frac{1}{1} = 1$. Therefore ϕ is a well defined injective ring homomorphism. □

Lemma 4.10. *Let $g : R \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for each $s \in S$ and $h : S^{-1}R \to B$ which takes $a/s \mapsto g(a)g(s)^{-1}$. If g is injective then h is injective.*

Proof.

$$\begin{aligned} Ker(h) &= \{\frac{a}{s} \in S^{-1}R | h(\frac{a}{s}) = 0\} \\ &= \{\frac{a}{s} \in S^{-1}R | g(a)g(s)^{-1} = 0\} \\ &= \{\frac{a}{s} \in S^{-1}R | g(a)g(s)^{-1}g(s) = 0\} \\ &= \{\frac{a}{s} \in S^{-1}R | g(a) = 0\}\end{aligned}$$

Since g is injective we have that $a = 0$. Hence $\text{Ker}(h) = \{0/s \in S^{-1}R\} = 0$. Therefore h is injective. □

Lemma 4.11. *Let $\phi : A \to B$ be a ring isomomorphism. Let $S \subset A$ be a multiplicatively closed subset and define $T = \phi(S)$. Then ϕ induces an isomorphism $S^{-1}A \to T^{-1}B$.*

Proof. $S^{-1}A = \{\frac{a}{s} : a \in A, s \in S\}$. $T^{-1}B = \{\frac{b}{t} : b \in B, t \in T\}$. We define

$$\begin{aligned} &\phi_* : S^{-1}A \to T^{-1}B \\ &\frac{a}{s} \mapsto \frac{\phi(a)}{\phi(s)}, \quad \phi(a) \in B, \phi(s) \in T\end{aligned}$$

Let $\frac{a}{s} = \frac{a'}{s'}$ in $S^{-1}A$. Then $(as' - a's)p = 0$ for some $p \in S$. We want to show $\frac{\phi(a)}{\phi(s)} = \frac{\phi(a')}{\phi(s')}$. This is true if we can find $q \in T$ such that $(\phi(a)\phi(s') -$

$\phi(a')\phi(s))q = 0$. Since $p \in S, \phi(p) \in T$ so we try $q = \phi(p)$.

$$\begin{aligned}(\phi(a)\phi(s') - \phi(a')\phi(s))\phi(p) &= (\phi(as') - \phi(a's))\phi(p) \\ &= \phi(as')\phi(p) - \phi(a's)\phi(p) \\ &= \phi(as'p) - \phi(a'sp) \\ &= \phi(as'p - a'sp) \\ &= \phi(0) \\ &= 0\end{aligned}$$

Hence ϕ_* is a well defined map.
Let $\frac{a}{s}, \frac{c}{s'} \in S^{-1}A$.

$$\begin{aligned}\phi_*(\frac{a}{s} + \frac{c}{s'}) = \phi_*(\frac{as' + cs}{ss'}) &= \frac{\phi(as' + cs)}{\phi(ss')} \\ &= \frac{\phi(as') + \phi(cs)}{\phi(s)\phi(s')} \\ &= \frac{\phi(as')}{\phi(s)\phi(s')} + \frac{\phi(cs)}{\phi(s)\phi(s')} \\ &= \frac{\phi(a)\phi(s')}{\phi(s)\phi(s')} + \frac{\phi(c)\phi(s)}{\phi(s)\phi(s')} \\ &= \frac{\phi(a)}{\phi(s)} + \frac{\phi(c)}{\phi(s')} \\ &= \phi_*(\frac{a}{s}) + \phi_*(\frac{c}{s'})\end{aligned}$$

$$\begin{aligned}\phi_*(\frac{a}{s} \cdot \frac{c}{s'}) = \phi_*(\frac{ac}{ss'}) &= \frac{\phi(ac)}{\phi(ss')} \\ &= \frac{\phi(a)\phi(c)}{\phi(s)\phi(s')} \\ &= \frac{\phi(a)}{\phi(s)} \cdot \frac{\phi(c)}{\phi(s')} \\ &= \phi_*(\frac{a}{s})\phi_*(\frac{c}{s'})\end{aligned}$$

$$\begin{aligned}\phi_*(\frac{1}{1}) &= \frac{\phi(1)}{\phi(1)} \\ &= \frac{1}{1}\end{aligned}$$

Hence ϕ_* is a ring homomorphism.

$$\begin{aligned}
\text{Ker } \phi_* &= \{\frac{a}{s} \in S^{-1}A : \phi_*(\frac{a}{s}) = 0 \text{ in } T^{-1}B\} \\
&= \{\frac{a}{s} \in S^{-1}A : \frac{\phi(a)}{\phi(s)} = 0 \text{ in } T^{-1}B\} \\
&= \{\frac{a}{s} \in S^{-1}A : \phi(a)q = 0 \text{ in } B \text{ for some } q \in T\} \\
&= \{\frac{a}{s} \in S^{-1}A : \phi(a)\phi(s') = 0 \text{ in } B \text{ for some } s' \in S\} \\
&= \{\frac{a}{s} \in S^{-1}A : \phi(as') = 0 \text{ in } B \text{ for some } s' \in S\} \\
&= \{\frac{a}{s} \in S^{-1}A : as' = 0 \text{ in } A \text{ for some } s' \in S\} \\
&= 0 \text{ in } S^{-1}A
\end{aligned}$$

Hence ϕ_* is injective.

Let $y \in T^{-1}B$. Can I find some $x \in S^{-1}A$ such that $\phi_*(x) = y$? Since $y \in T^{-1}B$, $y = \frac{b}{t}$ for some $b \in B$ and $t \in T$. Since ϕ is surjective $b = \phi(a)$ for some $a \in A$ and $t = \phi(s)$ for some $s \in S$. $\phi_*(\frac{a}{s}) = \frac{\phi(a)}{\phi(s)}$. Hence ϕ_* is surjective.

Therefore ϕ_* is a well defined, bijective ring homomorphism. □

4.0.1 Localization

Given a ring R and $x \in R$, we can form the localization $R_x = \{\frac{r}{x^n} : r \in R, n \geq 0\}$. This is a special case of $S^{-1}R$.

Lemma 4.12. *Let R be a ring, $x \in R$, and $\mathfrak{p} \subset R$ a prime ideal with $x \notin \mathfrak{p}$. Let $a, c \in R$ such that $\frac{a}{x^n} = \frac{b}{x^m}$ in R_x for $m, n \geq 0$. If $a \in \mathfrak{p}$ then $c \in \mathfrak{p}$.*

Proof. We are given $\frac{a}{x^n} = \frac{b}{x^m}$ in R_x. Hence $(ax^m - cx^n)x^q = 0$ for some $q \geq 0$. Therefore $ax^{m+q} - cx^{n+q} = 0$ from which we obtain $ax^{m+q} = cx^{n+q}$. We are given $a \in \mathfrak{p}$ hence $ax^{m+q} \in \mathfrak{p}$. Hence we have $cx^{n+q} \in \mathfrak{p}$. Since $x \notin \mathfrak{p}$, $x^n \notin \mathfrak{p}$ and $x^q \notin \mathfrak{p}$ hence $x^{n+q} \notin \mathfrak{p}$. $\mathfrak{p}$ is prime therefore c must be an element of $\mathfrak{p}$. □

Lemma 4.13. *Suppose $\mathfrak{p} \subset R$ is a prime ideal with $x \notin \mathfrak{p}$. Then $\mathfrak{p}_x \subset R_x$ is a prime ideal defined as $\mathfrak{p}_x = \{\frac{a}{x^n} : a \in \mathfrak{p}, n \geq 0\}$.*

Proof. First we check $\mathfrak{p}_x$ is an ideal in R_x. We want to check if the $0 \in R_x$ is in $\mathfrak{p}_x$. The $0 \in R_x$ is simply $\frac{0_R}{x^0} = \frac{0_R}{1}$. $\frac{0_R}{1}$ is also in $\mathfrak{p}_x$ since we have $\frac{0_R}{1}$ with $0_R \in \mathfrak{p}, 1 = x^0$.

Let $a, b \in \mathfrak{p}_x$. Then $a = \frac{a'}{x^n}$ and $b = \frac{b'}{x^m}$ for $a', b' \in \mathfrak{p}$ and $m, n \geq 0$.

$$
\begin{aligned}
a - b &= \frac{a'}{x^n} - \frac{b'}{x^m} \\
&= \frac{a'x^m - b'x^n}{x^{n+m}}
\end{aligned}
$$

$a'x^m, b'x^n \in \mathfrak{p}$ hence $a'x^m - b'x^n \in \mathfrak{p}$. $m + n \geq 0$. Therefore $a - b \in \mathfrak{p}_x$.
Let $r \in R_x, a \in \mathfrak{p}_x$. Then, $r = \frac{r'}{x^n}$ with $r' \in R, n \geq 0$ and $a = \frac{a'}{x^m}$ with $a' \in \mathfrak{p}, m \geq 0$.

$$
\begin{aligned}
ra &= \frac{r'}{x^n} \cdot \frac{a'}{x^m} \\
&= \frac{r'a'}{x^{n+m}}
\end{aligned}
$$

$r'a' \in \mathfrak{p}, n + m \geq 0$. Hence $ra \in \mathfrak{p}_x$.

Hence $\mathfrak{p}_x$ is an ideal in R_x.

Let $a, b \in R_x - \mathfrak{p}_x$. Then $a = \frac{c}{x^m}$ for some $c \in R - \mathfrak{p}, m \geq 0$ and $b = \frac{d}{x^l}$ for some $d \in R - \mathfrak{p}, l \geq 0$. Therefore $c, d \in R - \mathfrak{p}$. This means $c \notin \mathfrak{p}$ and $d \notin \mathfrak{p}$. Hence $cd \notin \mathfrak{p}$. Therefore $\frac{cd}{x^m x^l} \notin \mathfrak{p}_x$. Assume $1 \in \mathfrak{p}_x$. Then $\frac{1}{x^0} = \frac{a}{x^n}$ for some $a \in \mathfrak{p}$. Therefore $(1x^n - ax^0)x^q = 0$ for some $q \geq 0$. Hence $1x^{n+q} - ax^q = 0$. This means $1x^{n+q} = ax^q$. But $ax^q \in \mathfrak{p}$. This would mean $1x^{n+q} \in \mathfrak{p}$. We have $x^{n+q} \notin \mathfrak{p}$ which means $1 \in \mathfrak{p}$. This is a contradiction since $\mathfrak{p}$ is prime. Hence $1 \notin \mathfrak{p}_x$. Therefore $\mathfrak{p}_x$ is a prime ideal in R_x. □

We can also form localization of R by a prime ideal $\mathfrak{p}$ defined as $R_p = \{\frac{f}{g} : f \in R, g \in R - \mathfrak{p}\}$.
We can form localization of R_x by the prime ideal $\mathfrak{p}_x$ defined in Lemma 4.18. The localization of R_x by $\mathfrak{p}_x$ is defined as $(R_x)_{\mathfrak{p}_x} = \{\frac{a}{b} : a \in R_x, b \in R_x - \mathfrak{p}_x\}$. There is a natural isomorphism from $R_\mathfrak{p}$ to $(R_x)_{\mathfrak{p}_x}$.

Lemma 4.14. $R_\mathfrak{p} \cong (R_x)_{\mathfrak{p}_x}$

Proof. Let

$$
\begin{aligned}
\varphi : R_\mathfrak{p} &\to (R_x)_{\mathfrak{p}_x} \\
\frac{f}{g} &\mapsto \frac{\frac{f}{x^0}}{\frac{g}{x^0}}
\end{aligned}
$$

We will prove this is a well defined map, a ring homomorphism and bijective.
Let $\frac{f}{g} = \frac{h}{j}$ in $R_\mathfrak{p}$. Hence $(fj - hg)l = 0$ for some $l \in R - \mathfrak{p}$. $\varphi(f/g) = \frac{\frac{f}{x^0}}{\frac{g}{x^0}}$

$\varphi(h/j) = \frac{\frac{h}{x^0}}{\frac{j}{x^0}}$. We would like to show $\frac{\frac{f}{x^0}}{\frac{g}{x^0}} = \frac{\frac{h}{x^0}}{\frac{j}{x^0}}$. in other words can we find some $a \in R - \mathfrak{p}$ and $q \geq 0$ such that $(\frac{f}{x^0} \cdot \frac{j}{x^0} - \frac{g}{x^0} \cdot \frac{h}{x^0})\frac{a}{x^q} = 0$. We try $a = l$ and $q = 0$.

$$\begin{aligned}(\frac{f}{x^0} \cdot \frac{j}{x^0} - \frac{g}{x^0} \cdot \frac{h}{x^0})\frac{l}{x^0} &= (\frac{fj}{x^0} - \frac{gh}{x^0})\frac{l}{x^0} \\ &= (\frac{fj - gh}{x^0})\frac{l}{x^0} \\ &= \frac{fjl - ghl}{x^0} \\ &= \frac{0}{1} \\ &= 0\end{aligned}$$

Therefore φ is well defined.

$$\begin{aligned}\varphi(\frac{f}{g} + \frac{h}{j}) &= \varphi(\frac{fj + hg}{gj}) \\ &= \frac{\frac{fj+hg}{x^0}}{\frac{gj}{x^0}} \\ &= \frac{\frac{fj}{x^0} + \frac{hg}{x^0}}{\frac{gj}{x^0}} \\ &= \frac{\frac{f}{x^0}\frac{j}{x^0} + \frac{h}{x^0}\frac{g}{x^0}}{\frac{g}{x^0}\frac{j}{x^0}} \\ &= \frac{\frac{f}{x^0}\frac{j}{x^0}}{\frac{g}{x^0}\frac{j}{x^0}} + \frac{\frac{h}{x^0}\frac{g}{x^0}}{\frac{g}{x^0}\frac{j}{x^0}} \\ &= \frac{\frac{f}{x^0}}{\frac{g}{x^0}} + \frac{\frac{h}{x^0}}{\frac{j}{x^0}} \\ &= \varphi(\frac{f}{g}) + \varphi(\frac{h}{j})\end{aligned}$$

$$
\begin{aligned}
\varphi(\frac{f}{g}\frac{h}{j}) &= \varphi(\frac{fh}{gj})\\
&= \frac{\frac{fh}{x^0}}{\frac{gj}{x^0}}\\
&= \frac{\frac{f}{x^0}\frac{h}{x^0}}{\frac{g}{x^0}\frac{j}{x^0}}\\
&= \frac{\frac{f}{x^0}}{\frac{g}{x^0}} \cdot \frac{\frac{h}{x^0}}{\frac{j}{x^0}}\\
&= \varphi(\frac{f}{g})\varphi(\frac{h}{j})
\end{aligned}
$$

$\varphi(1/1) = \frac{\frac{1}{x^0}}{\frac{1}{x^0}}$. Therefore φ is a ring homomorphism.

$$
\begin{aligned}
Ker(\varphi) &= \{f/g : \varphi(f/g) = 0 \text{ in } (R_x)_{\mathfrak{p}_x}\}\\
&= \{f/g : \frac{\frac{f}{x^0}}{\frac{g}{x^0}} = 0 \text{ in } (R_x)_{\mathfrak{p}_x}\}\\
&= \{f/g : \frac{f}{x^0}t = 0 \text{ for some } t \in R_x - \mathfrak{p}_x\}\\
&= \{f/g : \frac{f}{x^0}\frac{h}{x^n} = 0 \in R_x, h \in R - \mathfrak{p}, n \geq 0\}\\
&= \{f/g : \frac{fh}{x^n} = 0\}\\
&= \{f/g : fhx^q = 0 \in R \text{ for some } q \geq 0\}
\end{aligned}
$$

Since $h \notin \mathfrak{p}$ and $x^q \notin \mathfrak{p}$, then $hx^q \notin \mathfrak{p}$. Therefore $hx^q \in R - \mathfrak{p}$. Hence $f/g = 0 \in R_{\mathfrak{p}}$. Therefore the $\text{Ker}(\varphi) = \{0\}$ and φ is injective.
Let $r \in (R_x)_{\mathfrak{p}_x}$. Can I find $r' \in R_{\mathfrak{p}}$ such that $\varphi(r') = r$? Given $r \in (R_x)_{\mathfrak{p}_x}$ we

can express r as $r = \frac{\frac{f}{x^m}}{\frac{g}{x^n}}$ with $f \in R, g \in R - \mathfrak{p}, m, n \geq 0$. Then,

$$r = \frac{\frac{f}{x^m}}{\frac{g}{x^n}}$$
$$r = \frac{\frac{f}{x^m} \cdot \frac{x^{n+m}}{x^0}}{\frac{g}{x^n} \cdot \frac{x^{n+m}}{x^0}}$$
$$r = \frac{\frac{fx^{n+m}}{x^m}}{\frac{gx^{m+n}}{x^n}}$$
$$r = \frac{\frac{fx^n x^m}{x^m}}{\frac{gx^m x^n}{x^n}}$$
$$r = \frac{\frac{fx^n}{x^0}}{\frac{gx^m}{x^0}}$$

Clearly $fx^n \in R$. We have $g \in R - \mathfrak{p}$. Therefore $g \notin \mathfrak{p}$. $x^m \notin \mathfrak{p}$. Hence $gx^m \notin \mathfrak{p}$. Therefore $gx^m \in R - \mathfrak{p}$. So we have found $r' = \frac{fx^m}{gx^n}$ such that $\varphi(r') = r$.

Hence $R_{\mathfrak{p}} \cong (R_x)_{\mathfrak{p}_x}$. □

Lemma 4.15. *Let R be a graded ring and $\mathfrak{p} \subseteq R$ a homogeneous prime ideal. Let $S = \{x \in R - \mathfrak{p} : x \text{ is homogeneous}\}$. S is multiplicatively closed.*

Proof. To show S is multiplicatively closed we need to show $1_R \in S$ and S is closed under multiplication. If $1 \in \mathfrak{p}$ then $\mathfrak{p} = R$ but we are given $\mathfrak{p}$ is a prime ideal hence $\mathfrak{p} \neq R$. Hence $1_R \in R - \mathfrak{p}$ where 1 is homogeneous of degree 0. Therefore $1 \in S$.
Let $a, b \in S$. Then $a, b \in R - \mathfrak{p}$. Hence $a \notin \mathfrak{p}$ and $b \notin \mathfrak{p}$. Therefore $ab \notin \mathfrak{p}$. Therefore $ab \in R - \mathfrak{p}$. a, b homogeneous hence ab is homogeneous.

Therefore S is a multiplicatively closed set. □

$S^{-1}R$ has a natural grading given by deg $\frac{f}{g}$ = deg $f -$ deg g for f homogeneous in R and $g \in S$.

Lemma 4.16. *Let R be a graded ring and $\mathfrak{p}$ a homogeneous prime ideal. Let $S = \{x \in R - \mathfrak{p} : x \text{ is homogeneous}\}$ Then the set of elements of degree 0, denoted $R_{(\mathfrak{p})}$, form a subring of the localization of R with respect to S.*

Proof. The set of elements of degree 0 are elements of the form $\frac{a}{x}, a \in R, x \in S$ with a, x homogeneous of same degree . Let $a_1, a_2 \in R_{(\mathfrak{p})}$. Then $a_1 = r_1/s_1$ and $a_2 = r_2/s_2$ with $r_1, r_2 \in R$, $s_1, s_2 \in S$, r_1, s_1 homogeneous of same degree and r_2, s_2 homogeneous of same degree. In other words deg r_1=deg s_1 and deg

$r_2 =$ deg s_2. Let degree of r_1, s_1 be c and degree r_2, s_2 be degree d.

$$\frac{r_1}{s_1} - \frac{r_2}{s_2} = \frac{r_1 s_2 - s_1 r_2}{s_1 s_2}$$

$s_1 s_2 \in S$ because S is multiplicatively closed. $r_1 s_2 - s_1 r_2 \in R$. deg $r_1 s_2 = c+d$. deg $s_1 r_2 = c+d$. deg $s_1 s_2 = c+d$. Hence $r_1 s_2 - s_1 r_2$ and $s_1 s_2$ are homogeneous of same degree. Hence $a_1 - a_2 \in R_{(\mathfrak{p})}$.

$$\frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$$

$s_1 s_2 \in S$, because S is multiplicatively closed. $r_1 r_2 \in R$. deg $r_1 r_2 = c + d$, deg $s_1 s_2 = c + d$. Hence $r_1 r_2, s_1 s_2$ are homogeneous of same degree. Therefore $a_1 a_2 \in R_{(\mathfrak{p})}$. $1 \in R_{(\mathfrak{p})}$. 1 is homogeneous of degree 0 and can be expressed as $1 = 1/1$ with $1 \in R, 1 \in S$. Hence $R_{(\mathfrak{p})}$ is a subring of $S^{-1}R$. □

Let R be a graded ring and $\mathfrak{p} \subseteq R$ a homogeneous prime ideal. Let $S \subset R$ where $S = \{x \in R - \mathfrak{p} : x \text{ is homogeneous}\}$. We can form homogeneous localization of R by $\mathfrak{p}$ defined as

$$R_{(\mathfrak{p})} = \{\frac{f}{g} : f \in R, g \in R - \mathfrak{p}, f \text{ and } g \text{ homogeneous of same degree in } S^{-1}R\}$$

Lemma 4.17. *Let $f \in R$ be homogeneous. Then $R_{(f)}$ is a subring of the localized ring R_f where $R_{(f)}$ is the set of elements of degree* 0.

Proof. Elements of R_f are of the form $a/f^n, a \in R, f^n \in T$ where $T = \{f^n | n \geq 0\}$. with a, f^n homogeneous of same degree.
Let $a/f^n, b/f^m \in R_{(f)}$. Let deg $f = d$. Then deg $a =$ deg $f^n = dn$ and deg $b =$ deg $f^m = dm$.

$$\frac{a}{f^n} - \frac{b}{f^m} = \frac{af^m - bf^n}{f^n f^m}$$

$af^m - bf^n \in R, f^n f^m \in T$. deg $af^m = dn + dm$. deg $bf^n = dm + dn$. deg $f^n f^m = dn + dm$. Hence $af^m - bf^n, f^n f^m$ are homogeneous of same degree. Therefore $\frac{a}{f^n} - \frac{b}{f^m} \in R_{(f)}$.

$$\frac{a}{f^n} \cdot \frac{b}{f^m} = \frac{ab}{f^n f^m}$$

$ab \in R, f^n f^m \in T$. deg $ab = dn + dm$. deg $f^n f^m = dn + dm$. Hence $ab, f^n f^m$ are homogeneous of same degree. Therefore $\frac{a}{f^n} \cdot \frac{b}{f^m} \in R_{(f)}$. $1 \in R_{(f)}$. 1 can be expressed as $1 = 1/1 = 1/f^0, 1 \in R, f^0 \in T$. $1, f^0$ are homogeneous of same degree namely 0. Hence $R_{(f)}$ is a subring of the localized ring R_f where $R_{(f)}$ is the set of elements of degree 0. □

Let $x \in R$ be homogeneous. The homogeneous localization of R by x is defined as $R_{(x)} = \{\frac{a}{x^n} : a \in R, n \geq 0, a, x^n$ homogeneous of same degree$\}$.

Lemma 4.18. *Let R be a graded ring. Suppose $\mathfrak{p} \subset R$ is a homogeneous prime ideal with x homogeneous and $x \notin \mathfrak{p}$. Then $\mathfrak{p}_{(x)} \subset R_{(x)}$ is a prime ideal defined as $\mathfrak{p}_{(x)} = \{\frac{a}{x^n} : a \in \mathfrak{p}, n \geq 0, a, x^n$ homogeneous of same degree$\}$.*

Proof. First we check $\mathfrak{p}_{(x)}$ is an ideal in $R_{(x)}$. We want to check if the $0 \in R_{(x)}$ is in $\mathfrak{p}_{(x)}$. The $0 \in R_{(x)}$ is simply $\frac{0_R}{x^0} = \frac{0_R}{1}$. $\frac{0_R}{1}$ is also in $\mathfrak{p}_{(x)}$ since we have $\frac{0_R}{1}$ with $0_R \in \mathfrak{p}, 1 = x^0$.

Let $a, b \in \mathfrak{p}_{(x)}$. Then $a = \frac{a'}{x^n}$ with $a' \in \mathfrak{p}, n \geq 0, a', x^n$, homogeneous of same degree r and $b = \frac{b'}{x^m}$ for $b' \in \mathfrak{p}, m \geq 0, b', x^m$ homogeneous of same degree t.

$$\begin{aligned} a - b &= \frac{a'}{x^n} - \frac{b'}{x^m} \\ &= \frac{a'x^m - b'x^n}{x^{n+m}} \end{aligned}$$

$a'x^m, b'x^n \in \mathfrak{p}$ hence $a'x^m - b'x^n \in \mathfrak{p}$. $a'x^m - b'x^n$ is homogeneous of degree $r + t$. x^{n+m} is homogeneous of degree $r + t$. $m + n \geq 0$. Therefore $a - b \in \mathfrak{p}_{(x)}$.

Let $s \in R_{(x)}, a \in \mathfrak{p}_{(x)}$. Then, $s = \frac{s'}{x^n}$ with $s' \in R, n \geq 0, s', x^n$ homogeneous of same degree r and $a = \frac{a'}{x^m}$ with $a' \in \mathfrak{p}, m \geq 0, a', x^m$ homogeneous of same degree t.

$$\begin{aligned} sa &= \frac{s'}{x^n} \cdot \frac{a'}{x^m} \\ &= \frac{s'a'}{x^{n+m}} \end{aligned}$$

$s'a' \in \mathfrak{p}, n + m \geq 0, s'a'$ homogeneous of degree $r + t$, x^{n+m} homogeneous of degree $r + t$. Hence $sa \in \mathfrak{p}_{(x)}$.

Hence $\mathfrak{p}_{(x)}$ is an ideal in $R_{(x)}$.

Let $a, b \in R_{(x)} - \mathfrak{p}_{(x)}$. Then $a = \frac{c}{x^m}$ for some $c \in R - \mathfrak{p}, m \geq 0, c, x^m$ homogeneous of same degree r and $b = \frac{d}{x^l}$ for some $d \in R - \mathfrak{p}, l \geq 0, d, x^l$ homogeneous of same degree t. Therefore $cd \in R - \mathfrak{p}$. This means $c \notin \mathfrak{p}$ and $d \notin \mathfrak{p}$. Hence $cd \notin \mathfrak{p}$. Therefore $ab = \frac{cd}{x^m x^l} \notin \mathfrak{p}_{(x)}$. Assume $1 \in \mathfrak{p}_{(x)}$. Then $\frac{1}{x^0} = \frac{a}{x^n}$ for some $a \in \mathfrak{p}$. Therefore $(1x^n - ax^0)x^q = 0$ for some $q \geq 0$. Hence $1x^{n+q} - ax^q = 0$. This means $1x^{n+q} = ax^q$. But $ax^q \in \mathfrak{p}$. This would mean $1x^{n+q} \in \mathfrak{p}$. We have

$x^{n+q} \notin \mathfrak{p}$ which means $1 \in \mathfrak{p}$. This is a contradiction since $\mathfrak{p}$ is prime. Hence $1 \notin \mathfrak{p}_{(x)}$. Therefore $\mathfrak{p}_{(x)}$ is a prime ideal in $R_{(x)}$. □

We define the localization of $R_{(x)}$ by $\mathfrak{p}_{(x)}$ as $(R_{(x)})_{\mathfrak{p}_{(x)}} = \{\frac{a}{b} : a \in R_{(x)}, b \in R_{(x)} - \mathfrak{p}_{(x)}\}$

Lemma 4.19. $R_{(\mathfrak{p})} \cong (R_{(x)})_{\mathfrak{p}_{(x)}}$

Proof. Let

$$\phi : R_{(\mathfrak{p})} \to (R_{(x)})_{\mathfrak{p}_{(x)}}$$
$$\frac{f}{g} \mapsto \frac{\frac{fg^{d-1}}{x^r}}{\frac{g^d}{x^r}}$$

with f, g of degree r and x has degree d. We will prove this is a well defined map, a ring homomorphism and bijective.

Let $\frac{f}{g} = \frac{h}{j}$ in $R_{(\mathfrak{p})}$. f, g of degree r, h, j of degree s. $(fj - hg)l = 0$ for some $l \in S$ with l of degree t. $\phi(f/g) = \frac{\frac{fg^{d-1}}{x^r}}{\frac{g^d}{x^r}}$ $\phi(h/j) = \frac{\frac{hj^{d-1}}{x^s}}{\frac{j^d}{x^s}}$. We would like to show $\frac{\frac{fg^{d-1}}{x^r}}{\frac{g^d}{x^r}} = \frac{\frac{hj^{d-1}}{x^s}}{\frac{j^d}{x^s}}$. In other words can we find some $a \in R_{(x)} - \mathfrak{p}_{(x)}$ such that $(\frac{fg^{d-1}}{x^r} \cdot \frac{j^d}{x^s} - \frac{g^d}{x^r} \cdot \frac{hj^{d-1}}{x^s})a = 0$. With $a \in R_{(x)} - \mathfrak{p}_{(x)}$, we can express a as $a = \frac{b}{x^q}$ with $b \in R - \mathfrak{p}, q \geq 0$. We try $b = l^d$ and $q = t$.

$$\begin{aligned}
(\frac{fg^{d-1}}{x^r} \cdot \frac{j^d}{x^s} - \frac{g^d}{x^r} \cdot \frac{hj^{d-1}}{x^s})\frac{l^d}{x^t} &= (\frac{fg^{d-1}j^d - g^d hj^{d-1}}{x^{r+s}})\frac{l^d}{x^t} \\
&= \frac{fg^{d-1}j^d l^d - g^d hj^{d-1}l^d}{x^{r+s+t}} \\
&= \frac{(g^{d-1}j^{d-1}l^{d-1})(fjl - ghl)}{x^{r+s+t}} \\
&= \frac{(g^{d-1}j^{d-1}l^{d-1})(0)}{x^{r+s+t}} \\
&= 0
\end{aligned}$$

Therefore ϕ is well defined.

$$
\begin{aligned}
\phi(\frac{f}{g}+\frac{h}{j}) &= \phi(\frac{fj+hg}{gj}) \\
&= \frac{\frac{(fj+hg)(gj)^{d-1}}{x^{r+s}}}{\frac{(gj)^d}{x^{r+s}}} \\
&= \frac{\frac{fg^{d-1}j^d+hg^dj^{d-1}}{x^{r+s}}}{\frac{g^dj^d}{x^{r+s}}} \\
&= \frac{\frac{fg^{d-1}j^d}{x^{r+s}}}{\frac{g^dj^d}{x^{r+s}}} + \frac{\frac{hj^{d-1}g^d}{x^{r+s}}}{\frac{g^dj^d}{x^{r+s}}} \\
&= \frac{\frac{fg^{d-1}j^d}{x^rx^s}}{\frac{g^dj^d}{x^rx^s}} + \frac{\frac{hj^{d-1}g^d}{x^rx^s}}{\frac{g^dj^d}{x^rx^s}} \\
&= \frac{\frac{fg^{d-1}}{x^r}}{\frac{g^d}{x^r}} + \frac{\frac{hj^{d-1}}{x^s}}{\frac{j^d}{x^s}} \\
&= \phi(\frac{f}{g}) + \phi(\frac{h}{j})
\end{aligned}
$$

$$
\begin{aligned}
\phi(\frac{f}{g}\frac{h}{j}) &= \phi(\frac{fh}{gj}) \\
&= \frac{\frac{fh(gj)^{d-1}}{x^{r+s}}}{\frac{(gj)^d}{x^{r+s}}} \\
&= \frac{\frac{fhg^{d-1}j^{d-1}}{x^rx^s}}{\frac{g^dj^d}{x^rx^s}} \\
&= \frac{\frac{fg^{d-1}hj^{d-1}}{x^rx^s}}{\frac{g^dj^d}{x^rx^s}} \\
&= \frac{\frac{fg^{d-1}}{x^r}\frac{hj^{d-1}}{x^s}}{\frac{g^d}{x^r}\frac{j^d}{x^s}} \\
&= \frac{\frac{fg^{d-1}}{x^r}}{\frac{g^d}{x^r}} \cdot \frac{\frac{hj^{d-1}}{x^s}}{\frac{j^d}{x^s}} \\
&= \phi(\frac{f}{g})\phi(\frac{h}{j})
\end{aligned}
$$

$$\phi(1/1) = \frac{\frac{1 \cdot 1^{d-1}}{x^0}}{\frac{1}{x^0}}$$
$$= \frac{\frac{1}{x^0}}{\frac{1}{x^0}}$$

Therefore ϕ is a ring homomorphism.

$$\begin{aligned}
Ker(\phi) &= \{f/g : \phi(f/g) = 0 \text{ in } (R_{(x)})_{\mathfrak{p}_{(x)}}\} \\
&= \{f/g : \frac{\frac{fg^{d-1}}{x^r}}{\frac{g^d}{x^r}} = 0 \text{ in } (R_{(x)})_{\mathfrak{p}_{(x)}}\} \\
&= \{f/g : \frac{fg^{d-1}}{x^r}(t) = 0 \text{ for some } t \in R_{(x)} - \mathfrak{p}_{(x)}\} \\
&= \{f/g : \frac{fg^{d-1}}{x^r}\frac{h}{x^q} = 0, h \in R - \mathfrak{p} \text{ of degree } dq, q \geq 0\} \\
&= \{f/g : \frac{fg^{d-1}h}{x^{r+q}} = 0\} \\
&= \{f/g : fg^{d-1}hx^n = 0 \text{ for some } n \geq 0\}
\end{aligned}$$

We have that $\frac{f}{g} = 0$ in $R_{(\mathfrak{p})}$ when $fs = 0$ for some $s \in S$. $g^{d-1} \in S, h \in S$ and $x^n \in S$. Hence $g^{d-1}hx^n \in S$. Therefore $\text{Ker}(\phi) = 0$ and ϕ is injective.

Let $y \in (R_{(x)})_{\mathfrak{p}_{(x)}}$. Can I find $\frac{f}{g} \in R_{(\mathfrak{p})}$ such that $\phi(\frac{f}{g}) = y$. Since $y \in (R_{(x)})_{\mathfrak{p}_{(x)}}$ it can be expressed as $y = \frac{\frac{h}{x^r}}{\frac{j}{x^s}}$ with $h \in R,$ of degree $dr, j \in R - \mathfrak{p},$ of degree ds.

We try $f = hx^s$ and $g = jx^r \notin \mathfrak{p}$.

$$\begin{aligned}
\phi\left(\frac{hx^s}{jx^r}\right) &= \frac{\frac{hx^s(jx^r)^{d-1}}{x^{dr+ds}}}{\frac{(jx^r)^d}{x^{dr+ds}}} \\
&= \frac{\frac{hx^s j^{d-1} x^{rd-r}}{x^{dr+ds}} \cdot \frac{x^r}{x^r}}{\frac{j^d x^{rd}}{x^{dr+ds}} \cdot \frac{x^s}{x^s}} \\
&= \frac{\frac{hx^s x^{rd} j^{d-1}}{x^{dr}x^{ds}x^r}}{\frac{j^d x^{rd} x^s}{x^{dr}x^{ds}x^s}} \\
&= \frac{\frac{hx^s j^{d-1}}{x^{ds}x^r}}{\frac{j^d x^s}{x^{ds}x^s}} \\
&= \frac{\frac{j^{d-1}x^s}{x^{ds}} \cdot \frac{h}{x^r}}{\frac{j^{d-1}x^s}{x^{ds}} \cdot \frac{j}{x^s}} \\
&= \frac{\frac{h}{x^r}}{\frac{j}{x^s}}
\end{aligned}$$

Hence ϕ is surjective.

Therefore $R_{(\mathfrak{p})} \cong (R_{(x)})_{\mathfrak{p}_{(x)}}$. □

Lemma 4.20. *Let B be an integral domain and $S = B - \{0\}$. Let $x \in S^{-1}B$. Then $I = \{s \in B|\ xs \in B\}$ is an ideal in B.*

Proof. Let $s, t \in I$. This means $xs \in B$ and $xt \in B$. Carrying out all calculations in the quotient field $S^{-1}B$ gives $xs + xt \in B$ and $x(s+t) \in B$. Also $s + t \in B$. Hence $s + t \in I$.
Let $s \in I$. Want $-s \in I$. Since $s \in I, xs \in B$. $xs \in S^{-1}B$ This means $-(xs) \in S^{-1}B$. $-(xs) = -(sx) = -sx = x(-s) \in S^{-1}B$. Hence $x(-s) \in B$. Hence $-s \in I$.
Let 0 be the additive neutral element in B. $x0 = 0 \in S^{-1}B$. Therefore $0 \in I$.
Let $b \in B$ and $s \in I$. Hence $xs \in B$ Since $b \in B$ and $xs \in B$, $bxs \in B$. $bxs \in S^{-1}B$. $bxs = xbs = x(bs)$. Therefore $x(bs) \in B$. Therefore $bs \in I$. This shows I is an ideal in B. □

Lemma 4.21. *If B is an integral domain, then B is equal to the intersection of its localizations at all maximal ideals. $B = \bigcap B_{\mathfrak{m}}$ in the quotient field of B where $\mathfrak{m} \subset B$ is a maximal ideal and $B_{\mathfrak{m}}$ is the localization of B at $\mathfrak{m}$.*

Proof. We have an injective map $B \to S^{-1}B$ where $S = B - \{0\}$ which sends $b \mapsto b/1$. By Lemma 4.10 we have an injective map $B_{\mathfrak{m}} \to S^{-1}B$ which sends $b/s \mapsto b/s, b \in B, s \in B - \mathfrak{m}$. Hence $B \subseteq B_{\mathfrak{m}}$ for each $B_{\mathfrak{m}}$. Hence $B \subseteq \bigcap B_{\mathfrak{m}}$
Let $x \in \bigcap B_{\mathfrak{m}}$. We can form all $s \in B$ such that $xs \in B$. This is the ideal in Lemma 4.20. $I = \{s \in B | xs \in B\}$. Want to show $I = B$. Assume

$I \neq B$. Pick one maximal ideal $\mathfrak{m} \subset B$. Then by assumption $x \in B_{\mathfrak{m}}$. Hence $x = a/t, a \in B, t \in B - \mathfrak{m}$. Therefore $xt = a$ and $xt \in B$. Therefore $t \in I$ but $t \notin \mathfrak{m}$. Hence $I \nsubseteq \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$. Hence $I = B$. This means $1 \in I$. If $1 \in I, x \in B$. Hence $\bigcap B_{\mathfrak{m}} \subseteq B$. Therefore $B = \bigcap B_{\mathfrak{m}}$. □

Lemma 4.22. *Let $f : A \to B$ be a ring isomorphism between integral domains. Then*

$$\tilde{f} : Q(A) \to Q(B)$$

$$\frac{c}{d} \mapsto \frac{f(c)}{f(d)}, \quad c, d \in A \text{ with } d \neq 0$$

is a well defined isomorphism of fields.

Proof. First we check $\frac{f(c)}{f(d)}$ is an element of $Q(B)$. Clearly $f(c) \in B$. f is an isomorphism, if $d \neq 0$ then $f(d) \neq 0$. Hence $\frac{f(c)}{f(d)}$ is an element of $Q(B)$.
Let $\frac{c}{d} = \frac{c'}{d'}$ in $Q(A)$. Then $(cd' - dc')u = 0$ for $u \neq 0$. Hence $cd' - dc' = 0$. Hence $cd' = c'd$ in A. From this equality we can conclude $f(cd') = f(c'd)$. Hence we have

$$\begin{aligned} f(cd') &= f(c'd) \\ f(c)f(d') &= f(c')f(d) \\ \frac{f(c)}{f(d)} &= \frac{f(c')}{f(d')} \end{aligned}$$

Hence $\tilde{f}$ is well defined.
Let $\frac{c}{d}, \frac{c'}{d'}$ in $Q(A)$.

$$\begin{aligned} \tilde{f}(\frac{c}{d} + \frac{c'}{d'}) &= \tilde{f}(\frac{cd' + c'd}{dd'}) \\ &= \frac{f(cd' + c'd)}{f(dd')} \\ &= \frac{f(cd') + f(c'd)}{f(dd')} \\ &= \frac{f(cd')}{f(dd')} + \frac{f(c'd)}{f(dd')} \\ &= \frac{f(c)f(d')}{f(d)f(d')} + \frac{f(c')f(d)}{f(d)f(d')} \\ &= \frac{f(c)}{f(d)} + \frac{f(c')}{f(d')} \\ &= \tilde{f}(\frac{c}{d}) + \tilde{f}(\frac{c'}{d'}) \end{aligned}$$

$$
\begin{aligned}
\tilde{f}(\frac{c}{d}\cdot\frac{c'}{d'}) &= \tilde{f}(\frac{cc'}{dd'}) \\
&= \frac{f(cc')}{f(dd')} \\
&= \frac{f(c)f(c')}{f(d)f(d')} \\
&= \frac{f(c)}{f(d)}\cdot\frac{f(c')}{f(d')} \\
&= \tilde{f}(\frac{c}{d})\cdot\tilde{f}(\frac{c'}{d'})
\end{aligned}
$$

$$
\begin{aligned}
\tilde{f}(\frac{1}{1}) &= \frac{f(1)}{f(1)} \\
&= \frac{1}{1}
\end{aligned}
$$

Therefore $\tilde{f}$ is a ring homomorphism.
f is a ring isomorphism hence it has an inverse map $g : B \to A$ which is a ring homomorphism. We guess an inverse map for $\tilde{f}$ namely

$$\tilde{g} : Q(B) \to Q(A)$$

$$\frac{u}{v} \mapsto \frac{g(u)}{g(v)}$$

Using the same proof we used for $\tilde{f}$, $\tilde{g}$ is a well defined map and a ring homomorphism. To check $\tilde{f}$ is an isomorphism we only have to check the composition of $\tilde{f}$ and $\tilde{g}$ are the respective identity maps.
Let $\frac{c}{d} \in Q(A)$. Then

$$
\begin{aligned}
(\tilde{g}\circ\tilde{f})(\frac{c}{d}) &= \tilde{g}(\tilde{f}(\frac{c}{d})) \\
&= \tilde{g}(\frac{f(c)}{f(d)}) \\
&= \frac{g(f(c))}{g(f(d))} \\
&= \frac{c}{d}
\end{aligned}
$$

Let $\frac{u}{v} \in Q(B)$. Then

$$
\begin{aligned}
(\tilde{f} \circ \tilde{g})(\frac{u}{v}) &= \tilde{f}(\tilde{g}(\frac{u}{v})) \\
&= \tilde{f}(\frac{g(u)}{g(v)}) \\
&= \frac{f(g(u))}{f(g(v))} \\
&= \frac{u}{v}
\end{aligned}
$$

Hence $Q(A) \cong Q(B)$. □

Lemma 4.23. *Let R be a graded ring which is also an integral domain and $\mathfrak{p} = (0)$. Then $R_{((0))}$ is a field.*

Proof. $R_{((0))} = \{\frac{a}{b}, a \in R, b \in R - (0), a, b$ homogeneous of same degree$\}$. Let $\frac{a}{b} \in R_{((0))}$ be a non-zero element, $b \in R - (0)$ and $a \neq 0$. Hence $a \in R - (0)$. Then $\frac{b}{a} \in R_{((0))}$ since $b \in R$ and non-zero. Also $a \in R - (0)$. $\frac{a}{b} \cdot \frac{b}{a} = 1$. Hence every non-zero element in $R_{((0))}$ has a multiplicative inverse. Therefore $R_{((0))}$ is a field. □

4.0.2 Local Rings

Definition 4.24. A ring is called a *local ring* if it has a unique maximal ideal.

Lemma 4.25. *Let R be a ring and $\mathfrak{m} \neq (1)$ an ideal of R such that every $x \in R - \mathfrak{m}$ is a unit in R. Then R is a local ring and $\mathfrak{m}$ its maximal ideal.*

Proof. We have that every element not in $\mathfrak{m}$ is a unit. Let $x \in R - \mathfrak{m}$ be a unit in R. If $\mathfrak{m}$ is not maximal then we can add elements not in $\mathfrak{m}$. But adding $x \in R - \mathfrak{m}$ to $\mathfrak{m}$ would generate R. Hence $\mathfrak{m}$ must be a maximal ideal. Let $\mathfrak{m}' \subseteq R$ be another maximal ideal. Since $\mathfrak{m}'$ is maximal $\mathfrak{m}' \neq R$. Hence $\mathfrak{m}'$ contains no units. Therefore $\mathfrak{m}' \subseteq \mathfrak{m}$. But $\mathfrak{m}'$ is maximal hence $\mathfrak{m}'$ must equal $\mathfrak{m}$. Therefore $\mathfrak{m}$ is a unique maximal ideal and R is a local ring. □

Lemma 4.26. *Let A be a ring and $\mathfrak{p}$ a prime ideal in A. Then $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p} \cdot A_{\mathfrak{p}}$*

Proof. $\mathfrak{p} \cdot A_{\mathfrak{p}}$ is the set of all elements a/b with $a \in \mathfrak{p}$ and $b \in A - \mathfrak{p}$.
The localization of A at $\mathfrak{p}$ is the set of all elements a/b with $a \in A$ and $b \in A - \mathfrak{p}$. $\mathfrak{p} \cdot A_{\mathfrak{p}} \subseteq A_{\mathfrak{p}}$. Let $a/b, c/d \in \mathfrak{p} \cdot A_{\mathfrak{p}}$. Then

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - cd}{bd}$$

$ad - cb \in \mathfrak{p}, bd \in A - \mathfrak{p}$. Hence

$\frac{a}{b} - \frac{c}{d} \in \mathfrak{p} \cdot A_{\mathfrak{p}}$. We have $0 = \frac{0}{1}$ since $0 \in \mathfrak{p}$ and $1 \in A - \mathfrak{p}$.Hence $0 \in \mathfrak{p} \cdot A_{\mathfrak{p}}$. Therefore $\mathfrak{p} \cdot A_{\mathfrak{p}}$ is an additive subgroup of $A_{\mathfrak{p}}$.

Let $a/b \in A_{\mathfrak{p}}$ and $a'/b' \in \mathfrak{p} \cdot A_{\mathfrak{p}}$. Then $a/b \cdot a'/b' = aa'/bb'$. $aa' \in \mathfrak{p}$ by definition of ideal. $bb' \in A - \mathfrak{p}$. Hence $aa'/bb' \in \mathfrak{p} \cdot A_{\mathfrak{p}}$. Therefore $\mathfrak{p} \cdot A_{\mathfrak{p}}$ is an ideal in $A_{\mathfrak{p}}$. To show $A_{\mathfrak{p}}$ is a local ring we need to show $\mathfrak{p} \cdot A_{\mathfrak{p}}$ is the unique maximal ideal in $A_{\mathfrak{p}}$. We must check that every element $x \in A_{\mathfrak{p}} - \mathfrak{p} \cdot A_{\mathfrak{p}}$ is a unit.
Let $x \in A_{\mathfrak{p}} - \mathfrak{p} \cdot A_{\mathfrak{p}}$. Then $x = \frac{a}{b}$ with $a \in A - \mathfrak{p}, b \in A - \mathfrak{p}$. But $\frac{b}{a} \in A_{\mathfrak{p}}$ with $b \in A, a \in A - \mathfrak{p}$. $\frac{a}{b} \cdot \frac{b}{a} = 1$. Therefore x is a unit. Hence by Lemma 4.25 $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p} \cdot A_{\mathfrak{p}}$ □

Lemma 4.27. *Let R be a graded ring and $\mathfrak{p}$ a prime ideal. Let $T = R - \mathfrak{p}$. $R_{(\mathfrak{p})}$ is a local ring with maximal ideal $(\mathfrak{p} \cdot T^{-1}R) \cap R_{(\mathfrak{p})}$.*

Proof. Elements in $(\mathfrak{p} \cdot T^{-1}R) \cap R_{(\mathfrak{p})}$ are the set $\{a/b | a \in \mathfrak{p}, b \in R - \mathfrak{p}, a, b$ homogeneous of same degree$\}$. Let $a/b, c/d \in (\mathfrak{p} \cdot T^{-1}R) \cap R_{(\mathfrak{p})}$. deg a=deg $b = r$. deg c=deg $d = q$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

$ad - bc \in \mathfrak{p}, bd \in R - \mathfrak{p}$. deg $ad = r + q$, deg $bc = r + q$ and deg $bd = r + q$. Hence $ad - bc, bd$ are homogeneous of same degree. Therefore $a/b - c/d \in (\mathfrak{p} \cdot T^{-1}R) \cap R_{(\mathfrak{p})}$. We have $0 = \frac{0}{b}$ for all b. We choose $b = 1$ $0 \in \mathfrak{p}$. $1 \in R - \mathfrak{p}$
Let $r \in R_{(\mathfrak{p})}$ and $a/b \in (\mathfrak{p} \cdot T^{-1}R) \cap R_{(\mathfrak{p})}$. $r = a'/b'$ with $a' \in R, b' \in R - \mathfrak{p}$ and deg $a' =$ deg $b' = q$.

$$\frac{a'}{b'} \cdot \frac{a}{b} = \frac{a'a}{b'b}$$

$a'a \in \mathfrak{p}$ by definition of an ideal. $b'b \in R - \mathfrak{p}$. deg $a'a = r + q$ deg$b'b = r + q$. Hence $a'a, b'b$ are homogeneous of same degree. Therefore $(\mathfrak{p} \cdot T^{-1}R) \cap R_{(\mathfrak{p})}$ is an ideal in $R_{(\mathfrak{p})}$.
Let $x \in R_{(\mathfrak{p})} - (\mathfrak{p} \cdot T^{-1}R) \cap R_{(\mathfrak{p})}$. Then $x = a/b, a \in R - \mathfrak{p}, b \in R - \mathfrak{p}, a, b$ homogeneous of same degree. But $b/a \in R_{(\mathfrak{p})}$ with $b \in R, a \in R - \mathfrak{p}$ and b, a homogeneous of same degree. $a/b \cdot b/a = 1$. Hence x is a unit. Therefore $R_{(\mathfrak{p})}$ is a local ring with maximal ideal $(\mathfrak{p} \cdot T^{-1}R) \cap R_{(\mathfrak{p})}$. □

5 Mappings between varieties

This section will discuss the mappings allowed between varieties.

5.1 Regular functions

Definition 5.1. Let K be an algebraically closed field. A variety over K is any affine, quasi-affine, projective, or quasi-projective variety.

Let Y be a quasi-affine variety in $\mathbf{A}^n$. We will consider maps $f : Y \to K$.

Definition 5.2. Let $Y \subseteq \mathbf{A}^n$ be a quasi-affine variety. A map $f : Y \to K$ is *regular* at a point $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$, and polynomials g, $h \in A = K[x_1, ..., x_n]$, such that h is nowhere zero on U, and $f = g/h$ on U. (we interpret the polynomials as functions on $\mathbf{A}^n$, hence on Y.) We say that f is regular on Y if it is regular at every point of Y.

Let Y be a quasi-projective variety in $\mathbf{P}^n$. We will consider maps $f : Y \to K$.

Definition 5.3. Let $Y \subseteq \mathbf{P}^n$ be a quasi-projective variety. A map $f : Y \to K$ is regular at a point $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$, and homogeneous polynomials $g, h \in S = K[x_0, ..., x_n]$, of the same degree, such that h is nowhere zero on U, and $f = g/h$ on U. We say that f is regular on Y if it is regular at every point.

Lemma 5.4. *Let Y be a quasi-affine or quasi-projective variety. If f and g are regular maps from Y to k, then $f - g$ and fg are regular. f/g where g is nowhere 0 is also regular.*

Proof. Case 1: Let Y be a quasi-affine variety. Let f and g be regular maps at a point $P \in Y$. This means there are open neighbourhoods U, V with $P \in U \subseteq Y$ and $P \in V \subseteq Y$, polynomials $f_1, f_2, g_1, g_2 \in A$ such that g_1 is nowhere 0 on U and $f = f_1/g_1$ on U. Also g_2 is nowhere 0 on V and $g = f_2/g_2$ on V.

$$\begin{aligned} f - g &= \frac{f_1}{g_1} - \frac{f_2}{g_2} \\ &= \frac{f_1 g_2 - g_1 f_2}{g_1 g_2} \text{ on } U \cap V \end{aligned}$$

Since $P \in U$ and $P \in V$, then $P \in U \cap V$. U and V are open in Y hence $U \cap V$ is open in Y. $f_1 g_2 - g_1 f_2 \in A$ and $g_1 g_2$ is nowhere 0 on $U \cap V$. Therefore $f - g$ is regular at P.

$$\begin{aligned} fg &= \frac{f_1}{g_1}\frac{f_2}{g_2} \\ &= \frac{f_1 f_2}{g_1 g_2} \text{ on } U \cap V \end{aligned}$$

Again since $P \in U$ and $P \in V$, then $P \in U \cap V$. U and V are open in Y hence $U \cap V$ is open in Y. $f_1 f_2 \in A$ and $g_1 g_2$ is nowhere 0 on $U \cap V$. Therefore fg is regular at P.

$$\begin{aligned} \frac{f}{g} &= \frac{\frac{f_1}{g_1}}{\frac{f_2}{g_2}} \\ &= \frac{f_1 g_1}{g_2 f_2} \text{ on } U \cap V \end{aligned}$$

Again $P \in U$ and $P \in V$, then $P \in U \cap V$. U and V are open in Y hence $U \cap V$ is open in Y. $f_1 g_1 \in A$. Since g is nowhere 0, g_2 is nowhere 0, then f_2 is nowhere 0. Hence $g_2 f_2$ is nowhere 0 on $U \cap V$. Therefore f/g is regular at P.

Case 2: Let Y be a quasi-projective variety. Let f and g be regular maps at a point $P \in Y$. This means there are open neighbourhoods U, V with $P \in U \subseteq Y$ and $P \in V \subseteq Y$, homogeneous polynomials $f_1, g_1 \in S$ such that g_1 is nowhere 0 on U and $f = f_1/g_1$ on U. Also there are homogeneous polynomials $f_2, g_2 \in S$ of same degree such that g_2 is nowhere 0 on V and $g = f_2/g_2$ on V.

$$\begin{aligned} f - g &= \frac{f_1}{g_1} - \frac{f_2}{g_2} \\ &= \frac{f_1 g_2 - g_1 f_2}{g_1 g_2} \text{ on } U \cap V \end{aligned}$$

Since $P \in U$ and $P \in V$, then $P \in U \cap V$. U and V are open in Y hence $U \cap V$ is open in Y. $f_1 g_2 - g_1 f_2$ and $g_1 g_2 \in S$ are homogeneous of same degree. $g_1 g_2$ is nowhere 0 on $U \cap V$. Therefore $f - g$ is regular at P.

$$\begin{aligned} fg &= \frac{f_1}{f_2} \frac{g_1}{g_2} \\ &= \frac{f_1 g_1}{f_2 g_2} \text{ on } U \cap V \end{aligned}$$

Again since $P \in U$ and $P \in V$, then $P \in U \cap V$. U and V are open in Y hence $U \cap V$ is open in Y. $f_1 g_1 \in A$ and $f_2 g_2$ is nowhere 0 on $U \cap V$. Therefore fg is regular at P.

$$\begin{aligned} \frac{f}{g} &= \frac{\frac{f_1}{f_2}}{\frac{g_1}{g_2}} \\ &= \frac{f_1}{f_2} \frac{g_2}{g_1} \text{ on } U \cap V \end{aligned}$$

Again $P \in U$ and $P \in V$, then $P \in U \cap V$. U is open and V are open in Y hence $U \cap V$ is open in Y. $f_1 g_2 \in A$. Since g is nowhere 0, g_2 is nowhere 0, then g_1 is nowhere 0. Hence $f_2 g_1$ is nowhere 0 on $U \cap V$. Therefore f/g is regular at P.

□

Lemma 5.5. *A regular map f is continuous, when K is identified with $\mathbf{A}^1$ in its Zariski topology.*

Proof. Let Y be a quasi-affine variety and $f : Y \to \mathbf{A}^1$ a regular function. Recall the topology on $\mathbf{A}^1$ is the Zariski topology. We have the supspace topology on Y. It is enough to show that f^{-1}(inverse image) of a closed set is closed. A closed set of $\mathbf{A}^1$ is a finite set of points or $\mathbf{A}^1$ itself. If the closed set is $\mathbf{A}^1$ then $f^{-1}(\mathbf{A}^1) = Y$. Y is closed in Y for the subspace topology if $Y = Y \cap C$ for some closed set in $\mathbf{A}^1$. But $Y = Y \cap \mathbf{A}^1$ so Y is closed in Y. It is sufficient to show that $f^{-1}(a) = \{P \in Y | f(P) = a\}$ is closed for any $a \in K$.

Let U be an open set such that $f = g/h$, with $g, h \in A$, and h nowhere 0 on U. Then $f^{-1}(a) \cap U = \{P \in U | f(P) = a\}$ which is $\{P \in U | g(P)/h(P) = a\}$. But $g(P)/h(P) = a$ if and only if $g(P)/h(P) - a = 0$ which gives $(g - ah)(P) = 0$. Hence $f^{-1}(a) \cap U = \{P \in U | (g - ah)(P) = 0\}$. $Z(g - ah) = \{P \in \mathbf{A}^n | (g - ah)(P) = 0\}$ which is closed in $\mathbf{A}^n$. So $f^{-1}(a) \cap U = Z(g - ah) \cap U$ is closed by the subspace topology on U. By Lemma 1.26 $f^{-1}(a)$ is closed in Y. □

Definition 5.6. Let $Y \subseteq \mathbf{A}^n$ be a quasi-affine variety. Let f be a regular function on Y. The zero set of f is defined as $Z(f) = \{P \in Y : f(P) = 0\}$

Lemma 5.7. *Let $Y \subseteq \mathbf{A}^n$ be a quasi-affine variety. Let $f : Y \to K$ be a regular function on Y. $Z(f)$ is closed in Y.*

Proof. $f^{-1}(\{0\}) = \{P \in Y : f(P) = 0\}$. $Z(f) = \{P \in Y : f(P) = 0\}$. Hence $Z(f) = f^{-1}(\{0\})$. $\{0\}$ is a closed subset. Since f is a regular function, f is continuous. Hence $Z(f)$ is closed in Y. □

Definition 5.8. Let Y be a quasi-affine variety. Let $f_1, \dots f_r$ be regular functions on Y. The zero set of $f_1, \dots f_r$ is defined as $Z(f_1, \dots f_r) = \{P \in Y : f_1(P) = f_2(P) = \cdots = f_r(P) = 0\}$

Lemma 5.9. *Let Y be a quasi-affine variety. Let $f_1, \dots f_r$ be regular functions on Y. $Z(f_1, \dots f_r)$ is closed.*

Proof. By Lemma 5.7 $Z(f_1), Z(f_2), \dots Z(f_r)$ are each closed in Y. Hence $Z(f_1) \cap Z(f_2) \cap \cdots \cap Z(f_r)$ is closed in Y. But $Z(f_1) \cap Z(f_2) \cap \cdots \cap Z(f_r) = \{f_1(P) = f_2(P) = \cdots = f_r(P) = 0\}$. This is precisely $Z(f_1, f_2, \dots, f_r)$. Hence $Z(f_1, f_2, \dots, f_r)$ is closed in Y. □

Lemma 5.10. *As is defined in Definition 5.3, g/h is a well defined function on U.*

Proof. Let g, h be homogeneous polynomials of degree d. Let $Q = (b_0, \dots b_n)$ be in the same equivalence class as P. Then Q=$(\lambda a_0, \dots \lambda a_n)$ for $\lambda \neq 0$.

$$\begin{aligned}\frac{g(\lambda a_0, \dots \lambda a_n)}{h(\lambda a_0, \dots \lambda a_n)} &= \frac{\lambda^d g(a_0, \dots a_n)}{\lambda^d h(a_0, \dots a_n)} \\ &= \frac{g(a_0, \dots a_n)}{h(a_0, \dots a_n)}\end{aligned}$$

Hence g/h is a well defined function on U when $h(a_0, \dots a_n) \neq 0$. □

Lemma 5.11. *f as defined in Definition 5.3 is continuous, when K is identified with $\mathbf{A}^1$ in its Zariski topology.*

Proof. $f : Y \to \mathbf{A}^1$. Recall the topology on $\mathbf{A}^1$ is the Zariski topology. We have the supspace topology on Y. It is enough to show that f^{-1}(inverse image) of a closed set is closed. A closed set of $\mathbf{A}^1$ is a finite set of points or $\mathbf{A}^1$ itself. We showed in Lemma 5.5 what happens when it is $\mathbf{A}^1$ so it is sufficient to show that $f^{-1}(a) = \{P \in Y | f(P) = a\}$ is closed for any $a \in K$. Let U be an open set such that $f = g/h$ on U, with $g, h \in S = K[x_0, \dots, x_n]$ homogeneous of degree d, and h nowhere 0 on U. Then $f^{-1}(a) \cap U = \{P \in U | f(P) = a\}$ which is $\{P \in U | g(P)/h(P) = a\}$. But $g(P)/h(P) = a$ if and only if $g(P)/h(P) - a = 0$. Taking $g - ah$ we see we have a homogenous polynomial hence we can look at $Z(g - ah)$. $Z(g - ah)$ is closed in $\mathbf{P}^n$. $Z(g - ah) = \{P \in \mathbf{P}^n | (g - ah)(P) = 0\}$. $Z(g - ah) \cap U = \{P \in U | (g - ah)(P) = 0\}$. By definition of subspace topology $Z(g - ah) \cap U$ is closed. But $f^{-1}(a) \cap U = \{P \in U | (g - ah)(P) = 0\}$. Hence $f^{-1}(a) \cap U = Z(g - ah) \cap U$. Therefore $f^{-1}(a) \cap U$ is closed in U. By Lemma 1.26 $f^{-1}(a)$ is closed in Y. $f^{-1}(a)$ is closed in Y. □

Definition 5.12. Let $Y \subseteq \mathbf{P}^n$ be a quasi-projective variety. Let f be a regular function on Y. The zero set of f is defined as $Z(f) = \{Q \in Y : f(Q) = 0\}$

Lemma 5.13. *Let $Y \subseteq \mathbf{P}^n$ be a quasi-projective variety. Let $f : Y \to K$ be a regular function on Y. $Z(f)$ is closed in Y.*

Proof. $f^{-1}(\{0\}) = \{P \in Y : f(P) = 0\}$. $Z(f) = \{P \in Y : f(P) = 0\}$. Hence $Z(f) = f^{-1}(\{0\})$. $\{0\}$ is a closed subset. Since f is a regular function, f is continuous. Hence $Z(f)$ is closed in Y. □

Remark. Similarly as in the quasi-affine case, if Y is a quasi-projective variety and $f_1, \dots f_r$ regular functions on Y then $Z(f_1, \dots f_r)$ is closed.

Lemma 5.14. *If f and g are regular functions on a variety Y, and $f = g$ on some nonempty open subset $U \subseteq Y$, then $f = g$ everywhere.*

Proof. We have that f and g are regular on Y. This means $f - g$ is regular on Y by Lemma 5.4 and therefore continuous. The set of points where $f - g = 0$ is $Z = \{P \in Y : f(P) - g(P) = 0\} \subseteq Y$. If Y is quasi-affine then by Lemma 5.7

this zero set is closed in Y. If Y is quasi-projective then by Lemma 5.13 this zero set is closed in Y.

By Theorem 1.46 every non-empty open subset of an irreducible space is dense. Hence by assumption U is dense in Y. Therefore the closure of U is all of Y. By definition, the closure of U is the intersection of all closed sets in Y that contain U. Hence each such closed set is all of Y. Z is one such closed set which contains U. Hence $Z = Y$. Therefore $f = g$ on Y. □

Lemma 5.15. *Let $U \subseteq \mathbf{P}^n$ be a quasi-projective variety and $f : U \to K$ a regular function on U. If $U' \subset U$ is open then $f_{|U'} : U' \to K$ is regular.*

Proof. Let $P \in U' \subseteq U$. Since f is regular at P there is an open neighbourhood $V \subseteq U$ with $P \in V$ and homogeneous polynomials $g, h \in K[x_0, \cdots, x_n]$ of same degree such that $h \neq 0$ on V and $f = \frac{g}{h}$ on V. To show $f_{|U'}$ is regular at P we need to find an open neighbourhood $V' \subseteq U'$ with $P \in V'$, homogeneous polynomials $g', h' \in K[x_0, \cdots, x_n]$ of same degree such that $h' \neq 0$ on V' and $f_{|U'} = \frac{g'}{h'}$ on V'. We guess $V' = U' \cap V$ and check it satisfies the conditions. $U' \cap V \subseteq U'$. $P \in U' \cap V$. We are given $f = \frac{g}{h}$ on V with $h \neq 0$ on V. But $f = \frac{g}{h}$ on $U' \cap V$ with $h \neq 0$ on $U' \cap V$ still holds true. Hence we have found an open neighbourhood that satisfies all conditions. Therefore $f_{|U'} : U' \to K$ is regular. □

Lemma 5.16. *Let X be a variety and $f : X \to K$ a map. If for each $P \in X$ there is an open $U \subseteq X$ with $P \in U$ such that $f_{|U} : U \to K$ is regular on U, then f is regular on X.*

Proof. Let X be an quasi-affine variety. Let $P \in X$ and let $U \subseteq X$ an open set with $P \in U$ such that $f_{|U} : U \to K$ is regular (on U). Hence there is an open $V \subseteq U$ with $P \in V$ such that $f_{|V} = g/h$ on V and $h \neq 0$ on V. To show f is regular at P we need an open subset of X that contains P such that $f = g'/h'$ on this subset and $h' \neq 0$ on this subset. This is satisfied since $V \subseteq X$ with $P \in V$ and f can be expressed as a quotient of polynomials on V with $f = g/h$. Hence f is regular on X.

Let X be an quasi-projective variety. Let $P \in X$ and let $U \subseteq X$ an open set with $P \in U$ such that $f_{|U} : U \to K$ is regular (on U). Hence there is an open $V \subseteq U$ with $P \in V$ such that $f_{|V} = g/h$ on V with $g, h \in K[x_0, \dots x_n]$ homogenous of same degree and $h \neq 0$ on V. To show f is regular at P we need an open subset of X that contains P such that $f = g'/h'$ on this subset with $g', h' \in K[x_0, \dots x_n]$ homogenous of same degree and $h' \neq 0$ on this subset. This is satisfied since $V \subseteq X$ with $P \in V$ and f can be expressed as a quotient of homogeneous polynomials on V with $f = g/h$. Hence f is regular on X. □

Lemma 5.17. *All constant functions are regular.*

Proof. Let Y be a quasi-affine variety. By definition a function $f : Y \to K$ is regular at a point $P \in Y$ if there exists an open neighbourhood U with $P \in U \subseteq Y$, $f = g/h$ on U, $g, h \in K[x_1, \dots x_n]$ h nowhere 0 on U. Let f be a constant function. An open neighbourhood for f is Y. f can be written as $f = \frac{g}{h}$ where $g = f$ and $h = 1$. Hence f is regular.

Let Y be quasi-projective variety. By definition a function $f : Y \to K$ is regular at a point $P \in Y$ if there exists an open neighbourhood U with $P \in U \subseteq Y$, $f = g/h$ on U, $g, h \in K[x_0, \dots x_n]$ are homogeneous. Let f be a constant function. It is homogeneous of degree 0. An open neighbourhood for f is Y. f can be written as $f = \frac{g}{h}$ where $g = f$ and $h = 1$. Hence f is regular. □

5.2 Morphisms

Definition 5.18. If X, Y are two varieties, a morphism $\phi : X \to Y$ is a continuous map such that for every open set $V \subseteq Y$, and every regular function $f : V \to K$, the function $f \circ \phi : \phi^{-1}(V) \to K$ is regular.

Lemma 5.19. *Let X, Y and Z be varieties. Let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms. Then the composition map $\psi \circ \phi : X \to Z$ is a morphism.*

Proof. By definition of morphism ϕ and ψ are continuous maps. By Lemma 1.32, $\psi \circ \phi$ is continuous. Let $V \subseteq Z$ be an open subset and f regular on V. Since ψ is a morphism $\psi^{-1}(V) \subseteq Y$ is open and $f \circ \psi : \psi^{-1}(V) \to K$ is regular. Since ϕ is a morphism, $\phi^{-1}(\psi^{-1}(V)) \subseteq X$ is open and $f \circ \psi \circ \phi : \phi^{-1}(\psi^{-1}(V)) \to K$ is regular. By Lemma 1.31 $\phi^{-1}(\psi^{-1}(V)) = (\psi \circ \phi)^{-1}(V)$. Hence $f \circ \psi \circ \phi : (\psi \circ \phi)^{-1}(V) \to K$ is regular. Therefore $\psi \circ \phi : X \to Z$ is a morphism. □

Example 5.20. Let $\varphi : \mathbf{A}^1 \to \mathbf{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Then φ defines a morphism.

Proof. By definition φ is continuous if and only if $\varphi^{-1}(C)$ is closed for all closed subsets $C \subseteq \mathbf{A}^2$. Let $C \subseteq \mathbf{A}^2$ be a closed subset. We would like to show $\varphi^{-1}(C) = \{t \in \mathbf{A}^1 | \varphi(t) \in C\}$ is closed in $\mathbf{A}^1$. The closed subsets in $\mathbf{A}^2$ are the algebraic sets. Therefore $C = Z(T)$ for some $T \subseteq K[x_1, x_2]$.

$$\begin{aligned}\varphi^{-1}(C) = \varphi^{-1}(Z(T)) &= \{t \in \mathbf{A}^1 | \varphi(t) \in Z(T)\} \\ &= \{t \in \mathbf{A}^1 | (t^2, t^3) \in Z(T)\} \\ &= \{t \in \mathbf{A}^1 | g(t^2, t^3) = 0 \text{ for all } g \in T\}\end{aligned}$$

We see $\varphi^{-1}(C)$ is the zero set $\{g(t^2, t^3) \in K[t] : g \in T\}$. i.e $Z(T')$ for some $T' \subseteq K[t]$. $Z(T')$ is closed in $\mathbf{A}^1$. Hence φ is continuous.

We have $\varphi : \mathbf{A}^1 \to \mathbf{A}^2$ defined by $t \mapsto (t^2, t^3)$. Let $V \subseteq \mathbf{A}^2$ be an open subset. Let $\varphi^{-1}(V) \subseteq \mathbf{A}^1$ be the inverse image of V under φ and $P \in \varphi^{-1}(V)$. Hence $\varphi(P) \in V$. Let $f : V \to K$ be a regular function. This means there is an open subset $U \subseteq V \subseteq \mathbf{A}^2$ with $\varphi(P) \in U \subseteq \mathbf{A}^2$, $f = g/h$ on U, $g, h \in K[x_1, x_2]$ and h nowhere 0. $f \circ \varphi : \varphi^{-1}(V) \to V \to K$. Since $U \subseteq V$ we can restrict our map as follows. $f \circ \varphi : \varphi^{-1}(U) \to U \to K$. This sends $t \mapsto (t^2, t^3) \mapsto g(t^2, t^3)/h(t^2, t^3)$.

$g(t^2,t^3)$,$h(t^2,t^3) \in K[t]$ and $h(t^2,t^3)$ is nowhere 0 on $\varphi^{-1}(U)$. Hence $f \circ \varphi$ is regular. Therefore φ defines a morphism. □

Lemma 5.21. *Let Y be a variety and $Y' \subseteq Y$ an open subset or irreducible closed subset. The inclusion map defined as*

$$\begin{aligned} i :&Y' \to Y \\ &y' \mapsto i(y') = y' \end{aligned}$$

is a morphism.

Proof. Let $W \subseteq Y$ be an open subset.

$$\begin{aligned} i^{-1}(W) &= \{P \in Y' : i(P) \in W\} \\ &= \{P \in Y' : P \in W\} \\ &= \{P \in Y' \cap W\} \\ &= Y' \cap W \end{aligned}$$

By the subspace topology on Y', $Y' \cap W$ is open in Y' hence $i^{-1}(W)$ is open in Y'. Therefore i is a continuous map.

Case 1: If Y is a quasi-affine variety.
Let $W \subset Y$ be an open subset and $f : W \to K$ regular on W. Let $Q \in i^{-1}(W)$. Then $i(Q) \in W$. Since f is regular for all points in W, there exists an open neighbourhood $O \subseteq W$ with $i(Q) \in O$ and $g, h \in K[x_1, \dots x_n]$ such that $h \neq 0$ on O and $f = g/h$ on O. Since i is continuous $i^{-1}(O)$ is open in Y'. By the subspace topology on $i^{-1}(W)$, $i^{-1}(O)$ is open in $i^{-1}(W)$. $Q \in i^{-1}(O)$. Hence we have found an open neighbourhood $i^{-1}(O) \subseteq i^{-1}(W)$ with $Q \in i^{-1}(O)$. We are given $h \neq 0$ on O. But $i^{-1}(O) = O$. Therefore $h \neq 0$ on $i^{-1}(O)$. Let $P \in i^{-1}(O)$. $(f \circ i)(P) = f(i(P)) = g(i(P))/h(i(P)) = g(P)/h(P)$. Hence $f \circ i$ is regular .

Case 2: If Y is a quasi-projective variety.
Let $W \subset Y$ be an open subset and $f : W \to K$ regular on W. Let $Q \in i^{-1}(W)$. Then $i(Q) \in W$. Since f is regular for all points in W, there exists an open neighbourhood $O \subseteq W$ with $i(Q) \in O$ and $g, h \in K[x_0, \dots x_n]$ homogeneous of same degree such that $h \neq 0$ on O and $f = g/h$ on O. Since i is continuous $i^{-1}(O)$ is open in Y'. By the subspace topology on $i^{-1}(W)$, $i^{-1}(O)$ is open in $i^{-1}(W)$. $Q \in i^{-1}(O)$. Hence we have found an open neighbourhood $i^{-1}(O) \subseteq i^{-1}(W)$ with $Q \in i^{-1}(O)$. We are given $h \neq 0$ on O. But $i^{-1}(O) = O$. Therefore $h \neq 0$ on $i^{-1}(O)$. Let $P \in i^{-1}(O)$. $(f \circ i)(P) = f(i(P)) = g(i(P))/h(i(P)) = g(P)/h(P)$. Hence $f \circ i$ is regular .

Therefore i is a morphism. □

Lemma 5.22. *Let X be a variety and $U \subseteq \mathbf{A}^n$ a non-empty open set. Let $j : U \hookrightarrow \mathbf{A}^n$ be the inclusion map. Let $\phi : X \to U$ be a map such that $j \circ \phi : X \to \mathbf{A}^n$ is continuous. Then ϕ is continuous.*

Proof. Let $O \subseteq U$ be open. By Lemma 1.18 O is open in $\mathbf{A}^n$. Therefore $(j \circ \phi)^{-1}(O)$ is open in X. By Lemma 1.31 $(j \circ \phi)^{-1}(O) = \phi^{-1}(j^{-1}(O))$. $\phi^{-1}(j^{-1}(O)) = \phi^{-1}(O)$. Hence $\phi^{-1}(O)$ is open in X. Therefore ϕ is continuous. □

Lemma 5.23. *Let X be a variety and $U \subseteq \mathbf{A}^n$ a non-empty open set. Let $j : U \hookrightarrow \mathbf{A}^n$ be the inclusion map. Let $\phi : X \to U$ be a map such that $j \circ \phi : X \to \mathbf{A}^n$ is a morphism. Then ϕ is a morphism.*

Proof. By Lemma 5.22 ϕ is continuous. Let $O \subseteq U \subseteq \mathbf{A}^n$ be open and $f : O \to K$ a regular function. By Lemma 1.18 O is also open in $\mathbf{A}^n$. We are given $j \circ \phi$ is a morphism. Hence for $O \subseteq \mathbf{A}^n$ open and $f : O \to K$ regular, then $f \circ j \circ \phi : (j \circ \phi)^{-1}(O) \to K$ is regular. By Lemma 1.31 $(j \circ \phi)^{-1}(O) = \phi^{-1}(j^{-1}(O))$. $\phi^{-1}(j^{-1}(O)) = \phi^{-1}(O)$. Let $Q \in \phi^{-1}(O)$. $(f \circ j \circ \phi)(Q) = (f \circ j)(\phi(Q)) = f(j(\phi(Q)) = f(\phi(Q)) = (f \circ \phi)(Q)$. Hence $f \circ j \circ \phi$ and $f \circ \phi$ are the same map. Therefore $f \circ \phi$ is regular. Hence ϕ is a morphism.

□

Lemma 5.24. *Let X, Y be varieties and $\phi : X \to Y$ a morphism. Let $X' \subseteq X$ be an irreducible closed subset. Then $\phi_{|X'} : X' \to Y$ is a morphism.*

Proof. We are given $\phi : X \to Y$ is a morphism. By Lemma 5.21 $i : X' \to X$ is a morphism. By Lemma 5.19, $\phi \circ i : X' \to Y$ is a morphism. Let $P \in X'$. $\phi_{|X'}(P) = \phi(P)$. $(\phi \circ i)(P) = \phi(i(P)) = \phi(P)$. Hence $\phi \circ i$ and $\phi_{|X'}$ are the same map. Therefore $\phi_{|X'}$ is a morphism. □

We now have a concept of defining an isomorphism of varieties.

5.2.1 Isomorphism of varieties

Definition 5.25. Let X, Y be varieties. An *isomorphism of varieties* $\phi : X \to Y$ is a morphism which admits an inverse morphism $\psi : Y \to X$ with $\psi \circ \phi = id_X$ where id_X is the identity map on X and $\phi \circ \psi = id_Y$ where id_Y is the identity map on Y. Note that an isomorphism is bijective and bicontinuous, but a bijective bicontinuous morphism need not be an isomorphism.

At the end of section 5, I will prove the note in Definition 5.25 using an example of a bijective bicontinuous morphism that is not an isomorphism.

5.3 Rings of functions

Definition 5.26. Let Y be a variety. We denote by $\mathcal{O}(Y)$ the ring of all regular functions on Y.

Let Y be a variety and $P \in Y$. Let $L = \{(U, f) : U \subseteq Y \text{ open }, P \in U, f : U \to K \text{ a regular function}\}$.

Lemma 5.27. *$(U, f) \sim (V, g) \leftrightarrow f = g$ on $U \cap V$ defines an equivalence relation on L.*

Proof. Reflexive: Let f be a regular map $U \to K$ on U. Then clearly $f = f$ on $U \cap U$. Hence $(U, f) \sim (U, f)$.

Symmetric: Let U, V be open sets and f, g regular maps such that $(U, f) \sim (V, g)$. Then $f = g$ on $U \cap V$. This also means $g = f$ on $V \cap U$. Hence $(V, g) \sim (U, f)$.

Transitive: Let $(U, f) \sim (V, g)$ and $(V, g) \sim (W, h)$. Then $f = g$ on $U \cap V$ and $g = h$ on $V \cap W$. Therefore $f = g = h$ on $U \cap V \cap W$. Hence $f = h$ on $U \cap V \cap W$. The set of points where $f = h$ or $f - h = 0$ is $Z = \{Q \in (U \cap W) : f(Q) - h(Q) = 0\} \subseteq U \cap W$. Y is irreducible and $U \cap W \subset Y$ is non-empty and open, by Theorem 1.46 $U \cap V$ is irreducible and dense. $U \cap V \cap W$ is a non-empty open subset of $U \cap W$, hence it is dense in $U \cap W$. This means $\overline{U \cap V \cap W} = U \cap W$. By definition, $\overline{U \cap V \cap W}$ in $U \cap W$ is the intersection of all closed sets in $U \cap W$ that contain $U \cap V \cap W$. Therefore $U \cap W$ is the intersection of all closed sets in $U \cap W$ that contain $U \cap V \cap W$. Hence each closed set is all of $U \cap W$. The only closed set in $U \cap W$ that contains $U \cap V \cap W$ is $U \cap W$. Z is one such closed set that contains $U \cap V \cap W$. Therefore $Z = U \cap W$. Hence $f = h$ on $U \cap W$.

Therefore $(U, f) \sim (V, g) \leftrightarrow f = g$ on $U \cap V$ defines an equivalence relation. □

Definition 5.28. A *germ* of a regular function on Y near a point P is an equivalence class in L. We denote the set of germs $\mathcal{O}_{P,Y}$.

Lemma 5.29. *Let Y be a variety. $U, V \subset Y$ open and f, g regular functions. $(U, f) + (V, g) = (U \cap V, f + g)$. $(U, f) + (V, g)$ is well defined on $\mathcal{O}_{P,Y}$ with the equivalence relation in Lemma 5.27.*

Proof. We have that $P \in U$ and $P \in V$. Hence $U \cap V$ is non empty. If $f : U \to K$, $g : V \to K$ are regular functions at P then f and g are regular at $P \in U \cap V$. Hence by Lemma 5.4 $f + g$ is regular on $U \cap V$. Let $(U, f) \sim (U', f')$ and $(V, g) \sim (V', g')$. Then $f = f'$ on $U \cap U'$ and $g = g'$ on $V \cap V'$. $(U, f) + (V, g) = (U \cap V, f + g)$. Since $f = f'$ on $U \cap U'$ and $g = g'$ on $V \cap V'$ we have that $f + g = f' + g'$ on $(U \cap U') \cap (V \cap V') = (U \cap V) \cap (U' \cap V')$. Hence $(U \cap V, f + g) \sim (U' \cap V', f' + g')$. Therefore $(U, f) + (V, g) = (U \cap V, f + g) \sim (U' \cap V', f' + g') = (U', f') + (V', g')$. Hence $(U, f) + (V, g)$ is well defined. □

Lemma 5.30. *Let $(U, f)(V, g) = (U \cap V, fg)$. $(U, f)(V, g)$ is well defined on $\mathcal{O}_{P,Y}$ with the equivalence relation in Lemma 5.27..*

Proof. We have that $P \in U$ and $P \in V$. Hence $U \cap V$ is non-empty. If $f : U \to K, g : V \to K$ are regular functions at P then f and g are regular at $P \in U \cap V$. Hence by Lemma 5.4 fg is regular on $U \cap V$. Let $(U, f) \sim (U', f')$ and $(V, g) \sim (V', g')$. Then $f = f'$ on $U \cap U'$ and $g = g'$ on $V \cap V'$. $(U, f)(V, g) = (U \cap V, fg)$. Since $f = f'$ on $U \cap U'$ and $g = g'$ on $V \cap V'$ we have that $fg = f'g'$ on $(U \cap U') \cap (V \cap V') = (U \cap V) \cap (U' \cap V')$. Hence $(U \cap V, fg) \sim (U' \cap V', f'g')$. Therefore $(U, f)(V, g) = (U \cap V, fg) \sim (U' \cap V', f'g') = (U', f')(V', g')$. Hence $(U, f)(V, g)$ is well defined. □

Lemma 5.31. *$\mathcal{O}_{P,Y}$ is a ring.*

Proof. We must check the following conditions hold.

(i) It is a commutative group with respect to addition.
(ii) Closed under Multiplication.
(iii) It is commutative with respect to multiplication
(iv) Multiplication is associative and has a unit element.
(v) For all $x, y, z \in R$ $(x + y)z = xz + yz$ and $z(x + y) = zx + zy$. We define addition as in Lemma 5.29 and multiplication as we defined in Lemma 5.30.

Proof of (i). Let $(U, f), (V, g)$ be in the set of equivalence classes. Then $(U, f) + (V, g) = (U \cap V, f + g)$. $U \cap V$ is open in Y, $f + g$ is regular in $U \cap V$. Hence the set is closed under addition. Let $(U, f), (V, g)$ and (W, h) be in the set of equivalence classes. Then

$$\begin{aligned}(U, f) + \big((V, g) + (W, h)\big) &= (U, f) + \big((V \cap W, g + h)\big)\\ &= \big(U \cap (V \cap W), f + (g + h)\big)\\ &= \big((U \cap V) \cap W, (f + g) + h\big)\\ &= (U \cap V, f + g) + (W, h)\\ &= \big((U, f) + (V, g)\big) + (W, h)\end{aligned}$$

Hence addition is associative. We guess $(Y, 0)$ as the additive neutral element,where 0 is the 0 function. Then $(U, f) + (Y, 0) = (U \cap Y, f + 0) = (U, f)$. So $(Y, 0)$ is indeed the additive neutral element. We try $(U, -f)$ as the additive inverse. $(U, f) + (U, -f) = (U \cap U, f + (-f)) = (U, 0) = (Y, 0)$ since $(U, 0) \sim (Y, 0)$.$(U, 0) \sim (Y, 0)$ since $0 = 0$ on $Y \cap U$. Let $(U, f), (V, g)$ be in the set of equivalence classes. $(U, f) + (V, g) = (U \cap V, f + g) = (V \cap U, g + f) = (V, g) + (U, f)$. Hence the set is commutative. Therefore $\mathcal{O}_{Y,P}$ is a commutative group.

Proof of (ii). Let $(U, f), (V, g)$ be in $\mathcal{O}_{Y,P}$. $(U, f)(V, g) = (U \cap V, fg)$. $U \cap V$ is open in Y, fg is regular in $U \cap V$. Hence the set is closed under multiplication.

Proof of (iii). Let $(U, f), (V, g)$ be in $\mathcal{O}_{Y,P}$. Then $(U, f)(V, g) = (U \cap V, fg) = (V \cap U, gf) = (V, g)(U, f)$.

Proof of (iv). Let $(U, f), (V, g), (W, h)$ be in $\mathcal{O}_{P,Y}$.

$$\begin{aligned}(U, f)((V, g)(W, h)) = (U, f)(V \cap W, gh) \\ =&(U \cap (V \cap W), f(gh)) \\ =&((U \cap V) \cap W), (fg)h) \\ =&(U \cap V, fg)(W, h) \\ =&((U, f)(V, g))(W, h)\end{aligned}$$

Hence multiplication is associative.
We try $(Y, 1)$ as our unit element where 1 is the constant function. $(Y, 1)(U, f) = (U \cap Y, f1) = (U, f)$. Hence we have a unit element.

Proof of (v). Let $(U, f), (V, g), (W, h)$ be in $\mathcal{O}_{P,Y}$.

$$\begin{aligned}(U, f)[(V, g) + (W, h)] = (U, f)(V \cap W, g + h) \\ =&(U \cap (V \cap W), f(g + h)) \\ =&(U \cap V \cap W, fg + fh) \\ =&((U \cap V) \cap (U \cap W), fg + fh) \\ =&(U \cap V, fg) + (U \cap W, fh) \\ =&(U, f)(V, g) + (U, f)(W, h)\end{aligned}$$

Since multiplication is commutative it follows that $[(V, g) + (W, h)](U, f) = (V, g)(U, f) + (W, h)(V, g)$.

Hence $\mathcal{O}_{P,Y}$ is a ring. □

Lemma 5.32. *$\mathcal{O}_{P,Y}$ is a local ring.*

Proof. Let W be the set of germs of regular functions that vanish at P. i.e $W = \{(U, f) | U \subseteq Y \text{ open and } f(P) = 0\}$. We claim this is the unique maximal ideal in $\mathcal{O}_{P,Y}$. First we show $W \neq \mathcal{O}_{P,Y}$. The unit element in $\mathcal{O}_{P,Y}$ is $(Y, 1)$. Since $f = 1$ is a constant function there is no P for which $f(P) = 0$. Hence $(Y, 1) \notin W$ and $W \neq \mathcal{O}_{P,Y}$. The elements of W are equivalence classes of pairs. W is a subgroup of $\mathcal{O}_{P,Y}$. Let $(V, g) \in \mathcal{O}_{P,Y}$. Then $(V, g)(U, f) = (V \cap U, gf)$. $V \cap U$ is open and $(gf)(P) = g(P)f(P) = g(P)0 = 0$. Hence $(V, g)(U, f) \in W$. Therefore W is an ideal.
From Lemma 4.25 we simply need to show that for (U, f) with $f(P) \neq 0$, (U, f) is a unit. Let $\frac{1}{f} : U\backslash Z(f) \to K$ and $Q \in U\backslash Z(f)$. Since f is nowhere 0 on $U - Z(f)$ then by Lemma 5.4 $\frac{1}{f}$ is regular. So we have $(U, f)((U\backslash Z(f), 1/f) = (U \cap (U\backslash Z(f), f1/f) = (U\backslash Z(f), 1)$. But $(U\backslash Z(f)) \cap Y = U\backslash Z(f)$. Hence $(U\backslash Z(f), 1) \sim (Y, 1)$ since $1 = 1$ on $U\backslash Z(f)$. □

Recall an ideal $\mathfrak{m}$ is maximal if and only if $R/\mathfrak{m}$ is a field. Since we have shown in Lemma 5.32 that $\mathcal{O}_{P,Y}$ is a local ring with maximal ideal W then $\mathcal{O}_{P,Y}/W$ is a field.

Lemma 5.33. *$\mathcal{O}_{P,Y}/W$ is isomorphic to K.*

Proof. Let $\phi : \mathcal{O}_{P,Y} \to K$ be defined by $(U, f) \mapsto f(P)$.First we check that ϕ is well defined. Let $(U, f) \sim (V, g)$. Then $f = g$ on $U \cap V$. $\phi(U, f) = f(P)$. $\phi(V, g) = g(P)$. But $(U, f) \sim (V, g)$ hence $f(P) = g(P)$. Therefore ϕ is well defined. Let $(U, f), (V, g)$ be in the ring of germs of regular functions at a point P. Then $\phi((U, f) + (V, g)) = \phi(U \cap V, f + g) = (f + g)P = f(P) + g(P) = \phi(U, f) + \phi(V, g)$.

$\phi((U, f)(V, g)) = \phi(U \cap V, fg) = (fg)P = f(P)g(P) = \phi(U, f)\phi(V, g)$.

$\phi(Y, 1) = 1(P) = 1$. Hence we have that ϕ is a ring homomorphism from $\mathcal{O}_{P,Y} \to K$. By [SL, Isomorphism Theorem, page 93] $\mathcal{O}_{P,Y}/$ Ker ϕ is isomorphic to Im(ϕ). Ker$(\phi) = \{(U, f) \in \mathcal{O}_{Y,P} | \phi(U, f) = 0\}$. ie Ker$(\phi) = \{(U, f) \in \mathcal{O}_{P,Y} | f(P) = 0\}$. This is indeed W. Hence $\mathcal{O}_{P,Y}/W$ is isomorphic to Im(ϕ). Let $a \in K$. Can we find (U, f) in $\mathcal{O}_{P,Y}$ such that $\phi(U, f) = a$? We try (Y, a) where a is viewed as a constant function. Constant functions are regular by Lemma 5.17. $P \in Y$ and $Y \subseteq Y$ is open. $\phi(Y, a) = a$. Hence ϕ is surjective. Therefore Im$(\phi) = K$. Thus $\mathcal{O}_{P,Y}/W$ is isomorphic to K. □

Lemma 5.34. *Let Y be a variety. Let $U \subseteq Y$ be open. Then $U - Z(f)$ is open in Y.*

Proof. $Z(f)$ is closed in U hence $U - Z(f)$ is open in U. $U - Z(f) \subseteq U$. By Lemma 1.18, $U - Z(f)$ is open in Y. Hence $U - Z(f)$ is open in Y. □

Let Y be a variety, $T = \{(U, f), U \subseteq Y, U$ non-empty and open $, f$ regular on $U\}$

Lemma 5.35. *$(U, f) \sim (V, g)$ iff $f = g$ on $U \cap V$ defines an equivalence relation on T.*

Proof. Reflexive: Let f be a regular map $U \to K$. Then $f = f$ on $U \cap U$. Hence $(U, f) \sim (U, f)$.
Symmetric: Let U, V be open subsets of Y and f, g regular maps such that $(U, f) \sim (V, g)$. Then $f = g$ on $U \cap V$. This means $g = f$ on $V \cap U$. Hence $(V, g) \sim (U, f)$.
Transitive: Let $(U, f) \sim (V, g)$ and $(V, g) \sim (W, h)$. Then $f = g$ on $U \cap V$ and $g = h$ on $V \cap W$. Therefore $f = g = h$ on $U \cap V \cap W$. Hence $f = h$ on $U \cap V \cap W$. The set of points where $f = h$ or $f - h = 0$ is $Z = \{Q \in (U \cap W) : f(Q) - h(Q) = 0\} \subseteq U \cap W$. Since Y is irreducible $U \cap W$ is irreducible and dense. $U \cap V \cap W$ is a non empty open subset of $U \cap W$ by Lemma 1.45, hence it is dense in $U \cap W$. This means $\overline{U \cap V \cap W} = U \cap W$.By definition, $\overline{U \cap V \cap W}$ in $U \cap W$ is the intersection of all closed sets in $U \cap W$ that contain $U \cap V \cap W$. Therefore $U \cap W$ is the intersection of all closed sets in $U \cap W$ that contain $U \cap V \cap W$. Hence each closed set is all of $U \cap W$. The only closed set in $U \cap W$ that contains $U \cap V \cap W$ is $U \cap W$. Z is one such closed set that contains $U \cap V \cap W$. Therefore $Z = U \cap W$. Hence $f = h$ on $U \cap W$.

□

The equivalence classes in T form a field called the *function field of* Y, denoted $K(Y)$. $K(Y) = \{[(U,f)] \mid (U,f) \in T\}$. Elements of $K(Y)$ are equivalence classes of pairs. We note here $K(Y)$ is standard notation for function field and K in this circumstance does not mean an algebraically closed field.

Lemma 5.36. *$K(Y)$ is a field.*

Proof. We define addition and multiplication as follows.

$$(U,f) + (V,g) = (U \cap V, f + g)$$

and

$$(U,f)(V,g) = (U \cap V, fg)$$

Addition and multiplication are well defined as we saw in Lemmas 5.29 and 5.30. The only difference here is that $U \cap V \neq \emptyset$ because $U \cap V$ is an open set in an irreducible space. We must check the following conditions hold.

(i) It is a commutative group with respect to addition.

(ii) Closed under Multiplication.

(iii) It is commutative with respect to multiplication.

(iv) Multiplication is associative and has a unit element.

(v) For all $x, y, z \in R$ $(x + y)z = xz + yz$ and $z(x + y) = zx + zy$.

(vi) For non-zero $(U,f) \in K(Y)$ there exists $(V,g) \in K(Y)$ such that $(U,f)(V,g)$ is the unit element.

The proof of (i)-(v) is similiar to Lemma 5.31 so we only need to prove (vi).

Let $(U,f) \in K(Y)$ with $f \neq 0$. This means there is a point $Q \in U$ such that $f(Q) \neq 0$. We restrict f to the open set $U - Z(f)$.

We have $\frac{1}{f} : (U - Z(f)) \to K$ and $Q \in U - Z(f)$. Since f is nowhere 0 by Lemma 5.4 $\frac{1}{f}$ is regular.

So we have $(U,f)((U - Z(f), 1/f) = (U \cap (U - Z(f), f1/f) = (U - Z(f), 1)$. But $(U - Z(f)) \cap Y = U - Z(f)$. Hence $(U - Z(f), 1) \sim (Y, 1)$ since $1 = 1$ on $U - Z(f)$. □

We have defined for any variety Y, $\mathcal{O}(Y)$, $\mathcal{O}_{Y,P}$ and $K(Y)$. Restricting functions we obtain natural maps from $\mathcal{O}(Y) \to \mathcal{O}_{Y,P} \to K(Y)$ which are injective and ring homomorphisms. So we can consider $\mathcal{O}(Y)$, $\mathcal{O}_{Y,P}$ as subrings of $K(Y)$. We will now show these maps are injective ring homomorphisms.

Lemma 5.37. *The map $\varphi : \mathcal{O}(Y) \to \mathcal{O}_{Y,P}$ defined by $\varphi(f) = [(Y,f)]$ for f regular on Y is injective and a ring homomorphism.*

Proof. Let $\varphi(f) = \varphi(g)$. Then $[(Y,f)] = [(Y,g)]$. This means $(Y,f) \sim (Y,g)$, hence $f = g$ on Y. Hence φ is injective.
Let f, g be regular.

$$\begin{aligned}\varphi(f+g) &= [(Y, f+g)] \\ &= [(Y,f)] + [(Y,g)] \\ &= \varphi(f) + \varphi(g)\end{aligned}$$

$$\begin{aligned}\varphi(fg) &= [(Y, fg)] \\ &= [(Y,f)][(Y,g)] \\ &= \varphi(f)\varphi(g)\end{aligned}$$

$\varphi(1) = [(Y, 1)]$.
Hence φ is a ring homomorphism. □

Lemma 5.38. *Let $\phi : \mathcal{O}_{Y,P} \to K(Y)$ be defined by $\phi[(U,f)] = [(U,f)]$ for U open, f regular on Y. Then ϕ is injective and a ring homomorphism.*

Proof. First we need to show ϕ is well defined. Let $[(U,f)] = [(V,g)]$. Then $(U,f) \sim (V,g)$. Hence $f = g$ on $U \cap V$. $\phi[(U,f)] = [(U,f)]$. $\phi[(V,g)] = [(V,g)]$. But $(U,f) \sim (V,g)$. Hence ϕ is well defined.

Let $\phi[(U,f)] = \phi[(V,g)]$. Then $[(U,f)] = [(V,g)]$ in $K(Y)$. This means $(U,f) \sim (V,g)$, therefore $f = g$ on $U \cap V$. Hence ϕ is injective.
Let $[(U,f)], [(V,g)] \in \mathcal{O}_{Y,P}$.

$$\begin{aligned}\phi([(U,f)] + [(V,g)]) &= \phi[(U \cap V, f+g)] \\ &= [(U \cap V, f+g)] \\ &= [(U,f)] + [(V,g)] \\ &= \phi[(U,f)] + \phi[(V,g)]\end{aligned}$$

.

$$\begin{aligned}\phi[(U,f)][(V,g)]) &= \phi[(U \cap V, fg)] \\ &= [(U \cap V, fg)] \\ &= [(U,f)][(V,g)] \\ &= \phi[(U,f)]\phi[(V,g)]\end{aligned}$$

$\phi[(Y,1)] = [(Y,1)]$.

Hence ϕ is a ring homomorphism. □

Lemma 5.39. $\mathcal{O}(Y) \subseteq \bigcap_{P \in Y} \mathcal{O}_{Y,P}$

Proof. We can consider $\mathcal{O}(Y)$ and $\bigcap_{P\in Y} \mathcal{O}_{Y,P}$ as subsets of $K(Y)$. Let $f \in \mathcal{O}(Y)$ and $P \in Y$. This means f is regular at P. Hence $f \in \mathcal{O}_{Y,P}$. Therefore $\mathcal{O}(Y) \subseteq \mathcal{O}_{Y,P}$. We can do this for arbitrary $P \in Y$. Hence $\mathcal{O}(Y) \subseteq \bigcap_{P\in Y} \mathcal{O}_{Y,P}$. □

Lemma 5.40. *Let $\varphi : X \to Y$ be a morphism. For each $P \in X, \varphi$ induces a ring homomorphism of local rings $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$.*

Proof. We will show that,

$$\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$$

which sends $[(V, f)] \mapsto [(\varphi^{-1}(V), f \circ \varphi)]$ is well defined.

Let $[(V, f)] \in \mathcal{O}_{\varphi(P),Y}$. Then $V \subseteq Y$ is open in Y, $\varphi(P) \in V$, and $f : V \to K$ is regular. We take $U = \varphi^{-1}(V)$. It is open in X and $P \in \varphi^{-1}(V)$. We take $g = f \circ \varphi : U \to V \to K$. It is regular. Let $[(V, f)], [(W, h)] \in \mathcal{O}_{\varphi(P),Y}$ such that $[(V, f)] = [(W, h)]$. Therefore $(V, f) \sim (W, h)$. This gives $f = h$ on $V \cap W$.
$\varphi_P^*[(V, f)] = [(\varphi^{-1}(V), f \circ \varphi)]$.
$\varphi_P^*[(W, h)] = [(\varphi^{-1}(W), h \circ \varphi)]$.

Let $P \in \varphi^{-1}(V \cap W) = \varphi^{-1}(V) \cap \varphi^{-1}(W)$. In particular $P \in \varphi^{-1}(V)$.
$(f \circ \varphi)(P) = f(\varphi(P))$. $(h \circ \varphi)(P) = h(\varphi(P))$. $\varphi(P) \in V \cap W$. Since $f = h$ on $V \cap W, f(\varphi(P)) = h(\varphi(P)$. Hence $h \circ \varphi = f \circ \varphi$ on $\varphi^{-1}(V) \cap \varphi^{-1}(W)$. Hence φ_P^* is well defined. We now check if this a ring homomorphism.
Let $[(V, f)], [(W, h)] \in \mathcal{O}_{\varphi(P),Y}$.

$$\begin{aligned}
\varphi_P^*((V, f) + (W, h)) &= \varphi_P^*(V \cap W, f + h) \\
&= (\varphi^{-1}(V \cap W), (f + h) \circ \varphi) \\
&= (\varphi^{-1}(V) \cap \varphi^{-1}(W), f \circ \varphi + h \circ \varphi) \\
&= (\varphi^{-1}(V), f \circ \varphi) + (\varphi^{-1}(W), h \circ \varphi) \\
&= \varphi_P^*(V, f) + \varphi_P^*(W, h)
\end{aligned}$$

$$\begin{aligned}
\varphi_P^*((V, f)(W, h)) &= \varphi_P^*(V \cap W, fh) \\
&= (\varphi^{-1}(V \cap W), (fh) \circ \varphi) \\
&= (\varphi^{-1}(V) \cap \varphi^{-1}(W), (f \circ \varphi)(h \circ \varphi)) \\
&= (\varphi^{-1}(V), f \circ \varphi)(\varphi^{-1}(W), h \circ \varphi) \\
&= \varphi_P^*(V, f)\varphi_P^*(W, h)
\end{aligned}$$

$$\begin{aligned}\varphi_P^*((Y,1)) &= (\varphi^{-1}(Y), 1\circ\varphi)\\ &= (X,1)\end{aligned}$$

Hence φ_P^* is a ring homomorphism. □

Lemma 5.41. *Let $\varphi : X \to Y$ be a map. Let $V \subseteq Y$. Then $\varphi(\varphi^{-1}(V)) = V \cap \varphi(X)$*

Proof. Let $y \in V \cap\ \varphi(X)$. Then $y \in V$ and $y \in \varphi(X)$. Since $y \in \varphi(X)$, $y = \varphi(P)$ for some $P \in X$. Also $y \in V$, therefore $\varphi(P) \in V$. Hence $P \in \varphi^{-1}(V)$. Therefore $y = \varphi(P) \in \varphi(\varphi^{-1}(V))$. Hence $V \cap \varphi(X) \subseteq \varphi(\varphi^{-1}(V))$.
Let $y \in \varphi(\varphi^{-1}(V))$. Then $y = \varphi(Q)$ for some $Q \in \varphi^{-1}(V)$. Hence $\varphi(Q) \in V$. Therefore $y \in V$. Since $\varphi^{-1}(V) \subseteq X$. $\varphi(Q) \in \varphi(X)$. Hence $y \in \varphi(X)$. Therefore $y \in V \cap \varphi(X)$. $\varphi(\varphi^{-1}(V)) \subseteq V \cap \varphi(X)$.
Hence $\varphi(\varphi^{-1}(V)) = V \cap \varphi(X)$. □

Lemma 5.42. *Let $\varphi(X)$ be dense in Y. Then $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$ is injective.*

Proof.

$$\begin{aligned}Ker(\varphi_P^*) &= \{(V,f) \in \mathcal{O}_{\varphi(P),Y} | \varphi_P^*(V,f) = 0\}\\ &= \{(V,f) \in \mathcal{O}_{\varphi(P),Y} | (\varphi^{-1}(V), f\circ\varphi) = 0\}\end{aligned}$$

Let $(V,f) \in \mathrm{Ker}(\varphi_P^*)$. Therefore we have $f \circ \varphi = 0$ on $\varphi^{-1}(V)$. Hence $f = 0$ on $\varphi(\varphi^{-1}(V))$. We see this by letting $Q \in \varphi(\varphi^{-1}(V)$. Then $Q = \varphi(R)$ for some $R \in \varphi^{-1}(V)$. Hence $f(Q) = f(\varphi(R)) = (f \circ \varphi)(R) = 0$. Given that $\varphi(X)$ is dense in Y and $\varphi(\varphi^{-1}(V)) = V \cap \varphi(X)$, Lemma 1.21 tells us $\varphi(\varphi^{-1}(V))$ is dense in V. Since $f = 0$ on $\varphi(\varphi^{-1}(V))$, by Lemma 5.14 $f = 0$ everywhere. i.e $f = 0$ on V. Therefore φ_P^* is injective. □

The next section will discuss the relationship of $\mathcal{O}(Y), \mathcal{O}_{P,Y}$ and $K(Y)$ to $A(Y)$ and $S(Y)$. We will show that for any affine variety Y, $A(Y) = \mathcal{O}(Y)$ but not in the case of a projective variety.

5.4 Rings of functions and coordinate rings

Theorem 5.43. *Let $Y \subseteq \mathbf{A}^n$ be an affine variety with affine coordinate ring $A(Y)$. Then,*

(a) $\mathcal{O}(Y) \cong A(Y)$

(b) for each $P \in Y$ let $\mathfrak{m}_P \subseteq A(Y)$ be the ideal of functions vanishing at P. Then $P \mapsto \mathfrak{m}_P$ gives a $1-1$ correspondence between the points of Y and the maximal ideals of $A(Y)$.

(c) For each P, $\mathcal{O}_{P,Y} \cong A(Y)_{\mathfrak{m}_P}$.

(d) $K(Y)$ is isomorphic to the quotient field of $A(Y)$.

Proof. Firstly we define a map $\alpha : A(Y) \to \mathcal{O}(Y)$. A polynomial $f \in K[x_1, \dots x_n]$ defines a regular function on $\mathbf{A}^n$ and hence on Y. So we obtain a ring homomorphism $\varphi : A \to \mathcal{O}(Y)$ defined by $f \mapsto f_{|Y}$. $\mathrm{Ker}(\varphi) = \{f \in A | f(P) = 0 \text{ for all } P \in Y\}$. This is simply $I(Y)$. So we have an injective homomorphism from $A/I(Y) \to \mathcal{O}(Y)$. Lemma 2.50 tells us there is a $1-1$ correspondence between points of Y and maximal ideals of A containing $I(Y)$. Passing to the quotient by $I(Y)$ as in Lemma 2.85, these maximal ideals correspond to maximal ideals of $A(Y)$. $P \mapsto I(P) \subseteq A \mapsto I(P)/I(Y) \subseteq A(Y)$.

For each P there is a natural map $\phi : A(Y)_{\mathfrak{m}_P} \to \mathcal{O}_{P,Y}$ that sends $j + I(Y)/l + I(Y) \mapsto (Y - Z(l), j/l)$. Since $A(Y) \to \mathcal{O}_{P,Y}$ is injective, ϕ is injective by Lemma 4.10.
ϕ is also surjective. To see this let $(U, f) \in \mathcal{O}_{P,Y}$. Then f is regular on U, there exists $V \subset U$ such that V is open with $P \in V \subseteq U$ and $f = g/h$, $g, h \in K[x_1, \dots, x_n]$ and h nowhere 0 on V. Let $Q \in V$. $f(Q) = g(Q)/h(Q)$ for all $Q \in V$ where $h(Q) \neq 0$. We want to find $a/s \in A(Y)_{\mathfrak{m}_P}$ such that $\phi(a/s) = (U, f)$. We try $a = g + I(Y), s = h + I(Y)$ with $a = g + I(Y) \in A(Y)$ and $s = h + I(Y) \in A(Y) - \mathfrak{m}_P$ since $h(P) \neq 0$ and hence $s \notin \mathfrak{m}_P$. $\phi(g + I(Y)/h + I(Y) = (Y - Z(h), g/h)$. From Lemma 5.27 we have $(U, f) \sim (V, f)$. We have $(Y - Z(h), g/h) \sim (V, g/h)$. Hence by the transitive property of an equivalence relation $(Y - Z(h), g/h) \sim (U, f)$. Therefore ϕ is surjective. Hence for each P, $\mathcal{O}_{P,Y} \cong A(Y)_{\mathfrak{m}_P}$.

In a proof similar to (c), we have an injective map from the quotient field of $A(Y)$ which we denote $S^{-1}R$, to $K(Y)$ since $g : A(Y) \to K(Y)$ is injective and $g(s)$ is a unit in $K(Y)$ for each $s \in S = A(Y) - \{0\}$. This is because every field without 0 is a multiplicative group. All elements in this group are units. The map from the quotient field of $A(Y)$ is also surjective. To see this let $(U, f) \in K(Y)$. Then f is regular on U, there exists $V \subset U$ such that V is open in U and $f = g/h$, $g, h \in K[x_1, \dots, x_n]$ with h nowhere 0 on V. Let $Q \in V$. $f(Q) = g(Q)/h(Q)$ for all $Q \in V$ where $h(Q) \neq 0$. We want to find $a/s \in S^{-1}R$ such that $\phi(a/s) = (U, f)$. We try $a = g + I(Y), s = h + I(Y)$ with $a = g + I(Y) \in A(Y)$ and $s = h + I(Y) \in A(Y) - \{0\}$. $h + I(Y) \neq 0 + I(Y)$. If $h + I(Y) = 0 + I(Y)$, this means $h \in I(Y)$. But $h \notin I(Y)$ since $h(P) \neq 0$. Hence we have $\phi(g + I(Y)/h + I(Y)) = (Y - Z(h), g/h) \sim (U, f)$. Hence $K(Y)$ is isomorphic to the quotient field of $A(Y)$. Now we can prove (a). By Lemma 5.39 $\mathcal{O}(Y) \subseteq \bigcap \mathcal{O}_{P,Y}$. From (b) and (c) we have $A(Y) \subseteq \mathcal{O}(Y) \subseteq \bigcap A(Y)_{\mathfrak{m}}$ where $\mathfrak{m}$ is maximal ideal. From Lemma 4.21 we have that $A(Y) = \bigcap A(Y)_{\mathfrak{m}}$.

Hence $A(Y) = \mathcal{O}(Y)$. □

Lemma 5.44. *Let $U_i^n \subseteq \mathbf{P}^n$: be the open set defined by $x_i \neq 0$. Then $\phi_i^n : U_i^n \to \mathbf{A}^n$ is an isomorphism of varieties.*

Proof. We will first show ϕ_i^n is a morphism. Recall ϕ_i^n is the map defined as

$$\begin{aligned}\phi_i^n : & U_i^n \to \mathbf{A}^n \\ & P \mapsto Q\end{aligned}$$

where $P = (a_0, \dots a_n) \in U_i^n$, $Q = \phi_i^n(P) = (a_0/a_i, \dots a_{i-1}/a_i, a_{i+1}/a_i, \dots a_n/a_i)$ with a_i/a_i omitted.
By Lemma 3.56 ϕ_i^n is a bijective, continuous map whose inverse map ψ_i^n is continuous. Let $V \subseteq \mathbf{A}^n$ be an open subset and $f : V \to K$ a regular function. Let $Q \in (\phi_i^n)^{-1}(V)$. Then $\phi_i^n(Q) \in V$. Since f is regular on V, there exists an open neighbourhood $W \subseteq V$ with $\phi_i^n(Q) \in W$ and polynomials $g, h \in K[x_1, \dots, x_n]$ such that $h \neq 0$ on W and $f = g/h$ on W.

Given that $\phi_i^n(Q) \in W$ we have that $Q \in (\phi_i^n)^{-1}(W) \subseteq (\phi_i^n)^{-1}(V)$. By Lemma 1.18, W is open in $\mathbf{A}^n$. ϕ_i^n is continuous. Hence $(\phi_i^n)^{-1}(W)$ is open in U_i^n. By the subspace topology on $(\phi_i^n)^{-1}(V)$, $(\phi_i^n)^{-1}(W)$ is open in $(\phi_i^n)^{-1}(V)$. Therefore, for a point $Q \in (\phi_i^n)^{-1}(V)$ we have found an open neighbourhood $(\phi_i^n)^{-1}(W) \subseteq (\phi_i^n)^{-1}(V)$ which contains Q.

$f \circ \phi_i^n$ is a function on the open set $(\phi_i^n)^{-1}(V)$. We have $f = g/h$ on W. We can put g and h through β as follows.
$\beta(g) = x_i^e g(x_0/x_i, \dots, x_{i-1}/x_i, x_{i+1}/x_i, \dots, x_n/x_i)$ homogeneous of degree e. $\beta(h) = x_i^d h(x_0/x_i, \dots x_{i-1}/x_i, x_{i+1}/x_i, \dots, x_n/x_i)$ homogeneous of degree d. Let $c = \max \text{degree}(d, e)$.
Put $g' = x_i^c g(x_0/x_i, \dots, x_n/x_i)$ and $h' = x_i^c h(x_0/x_i, \dots, x_n/x_i)$. Then g', h' are homogeneous of degree c. Let $Q' \in (\phi_i^n)^{-1}(W)$. $Q' = [q_0 : q_1 : \dots q_{i-1} : 1 : q_{i+1} \cdots : q_n]$.

$$\begin{aligned}(f \circ \phi_i^n)(Q')) &= f(\phi_i^n(Q')) \\ &= f(q_0/1, \dots q_{i-1}/1, q_{i+1}/1, \dots q_n/1) \\ &= f(q_0, \dots q_{i-1}, q_{i+1}, \dots q_n) \\ &= \frac{g(q_0 \dots q_n)}{h(q_0 \dots q_n)} \\ &= \frac{g'(Q')}{h'(Q')}\end{aligned}$$

By Lemma 5.4 $f \circ \phi_i^n$ is regular.

Hence ϕ_i^n is a morphism.
We now check if the map ψ_i^n is a morphism. This map is defined by

$$\begin{aligned}\psi_i^n : \mathbf{A}^n &\to U_i^n \\ (b_1, \dots b_n) &\mapsto (b_1, \dots b_i, 1, b_{i+1} \dots, b_n)\end{aligned}$$

Let j be a regular function on an open set $U \subseteq U_i^n$. Then for every $P \in U$ there is an open neighbourhood Z with $P \in Z \subseteq U$ and homogeneous polynomials $l, m \in K[x_0 \dots x_n]$ such that $j = l/m$ on Z. Since ψ_i^n is continuous then the inverse image of an open set is open. Let $S \in (\psi_i^n)^{-1}(U)$ be a point. Then $\psi_i^n(S) \in U$. Therefore there exists an open neighbourhood Z with $\psi_i(S) \in Z \subseteq U$ and homogeneous polynomials $l, m \in K[x_0, \dots, x_n]$ such that $j = l/m$ on Z. Hence $(\psi_i^n)^{-1}(Z)$ is an open neighbourhood of S in $(\psi_i^n)^{-1}(U)$. $j \circ \psi_i^n$ is a function on the open set $(\psi_i^n)^{-1}(U)$.

We have $j = l/m$ on Z where $l, m \in K[x_0, \dots, x_n]$ and m is not zero on Z. We can put l and m through α as follows.
$\alpha(l) = l(b_1, \dots b_i, 1, b_{i+1}, \dots, b_n)$. $\alpha(m) = m(b_1, \dots, b_i, 1, b_{i+1} \dots b_n)$. Then

$$\begin{aligned}\frac{l(b_1, \dots, b_i, 1, b_{i+1}, \dots, b_n)}{m(b_1, \dots, b_i, 1, b_{i+1}, \dots b_n)} &= j(b_1, \dots, b_i, 1, b_{i+1}, \dots, b_n) \\ &= j(\psi_i^n(b_1, \dots b_n)) \\ &= (j \circ \psi_i^n)(b_1, \dots, b_n)\end{aligned}$$

Hence ψ_i^n is a morphism.
Let $P = [a_0 : a_1 :, \dots, a_n]$.

$$\begin{aligned}(\psi_i^n \circ \phi_i^n)(a_0, \dots a_n) &= \psi_i^n(a_0/a_i, \dots a_{i-1}/a_i, a_{i+1}/a_i \dots : a_n/a_i) \\ &= (a_0/a_i : \dots a_{i-1}/a_i, 1, a_{i+1}/a_i, \dots, a_n/a_i) \\ &\sim (a_0 : a_1 : \dots : a_{i-1} : a_i : a_{i+1} : \dots, a_n).\end{aligned}$$

Let $Q = (b_1, \dots, b_n)$.

$$\begin{aligned}(\phi_i^n \circ \psi_i^n)(b_1, \dots b_n) &= \phi_i^n(b_1 : \dots : b_i, 1 : b_{i+1} \dots : b_n) \\ &= (b_1, \dots, b_n)\end{aligned}$$

Hence ϕ_i^n is an isomorphism. □

We have $S = K[x_0, \dots, x_n]$ is a graded ring. Let $\mathfrak{p}$ be a homogeneous prime ideal of S then from Lemma 4.16, $S_{(\mathfrak{p})}$ is the subring of elements of degree 0 in the localization of S with respect to the multiplicative subset T consisting of the homogeneous elements of S not in $\mathfrak{p}$. We denote this localization ring as $T^{-1}S$. By Lemma 4.27 $S_{(\mathfrak{p})}$ is a local ring with maximal ideal $(\mathfrak{p} \cdot T^{-1}S) \cap S_{(\mathfrak{p})}$. In particular if S is an integral domain and we have $\mathfrak{p} = (0)$, $S_{((0))}$ is a field by Lemma 4.23. Also if $f \in S$ is a homogeneous element then by Lemma 4.17, $S_{(f)}$ is a subring of the localized ring S_f where $S_{(f)}$ is the set of elements of degree 0.

Lemma 5.45. *Let $f : R \to S$ and $g : R \to S$ be ring homomorphisms. Then $D = \{r \in R : f(r) = g(r)\}$ is a subring of R.*

Proof. $\{r \in R : f(r) = g(r)\} \subseteq R$. Let $r, s \in D$. Then $f(r) = g(r)$ and $f(s) = g(s)$. Since f and g are ring homomorphisms $f(r-s) = f(r) - f(s) = g(r) - g(s) = g(r-s)$. Hence $r - s \in D$. $f(rs) = f(r)f(s) = g(r)g(s) = g(rs)$. Hence $rs \in D$. $f(1) = 1$ by definition of a ring homomorphism. $g(1) = 1$ by definition of a ring homomorphism. Hence $f(1) = g(1)$ and $1 \in D$. Therefore D is a subring of R. □

Let $U_i^n \subseteq \mathbf{P}^n$ be the open set defined by $x_i \neq 0$. So far we have defined an isomorphism of varieties,

$$\begin{aligned} \phi_i^n : U_i^n &\to \mathbf{A}^n \\ (a_0 : \cdots : a_n) &\mapsto (a_0/a_i, \ldots a_{i-1}/a_i, a_{i+1}/a_i \ldots a_n/a_i) \end{aligned}$$

We have defined a morphism

$$\begin{aligned} \psi_i^n : &\mathbf{A}^n \to \mathbf{P}^n \\ (b_1, \ldots b_n) &\mapsto (b_1, \ldots, b_i, 1, b_{i+1}, \ldots, b_n) \end{aligned}$$

We have a ring homomorphism.

$$\begin{aligned} (\phi_i^n)^* : K[y_1, \ldots, y_n] &\to K[x_0, \ldots, x_n]_{(x_i)} \\ y_j &\mapsto \begin{cases} \frac{x_{j-1}}{x_i} \text{ if } 1 \leq j \leq i \\ \frac{x_j}{x_i} \text{ if } i+1 \leq j \leq n \end{cases} \end{aligned}$$

We have $U_i^n \subseteq \mathbb{P}^n$ an open set such that $x_i \neq 0$. We let $Y_i = Y \cap U_i^n$. Since $U_i^n \cong \mathbb{A}^n$ we can consider Y_i as the affine variety $\phi_i^n(Y_i)$.

Lemma 5.46. *For fixed $P \in U_i^n$ the set $D = \{f \in K[y_1, \ldots, y_n] : f(\phi_i^n(P)) = ((\phi_i^n)^* f)(P)\}$ is a subring of $K[y_1, \ldots, y_n]$.*

Proof. By Lemma 5.45, D is a subring of $K[y_1, \ldots, y_n]$. □

Corollary 5.47. D is a subring of $K[y_1, \ldots, y_n]$. In the case where $f = y_j \in K[y_1, \ldots, y_n]$, then $D = K[y_1, \ldots, y_n]$.

Proof. Let $f = y_j \in K[y_1, \ldots, y_n]$ and $P = (a_0, \ldots, a_n) \in U_i^n$. We may assume 1 is in the i^{th} position. Then $f(\phi_i^n(P)) = f(a_0/1 \ldots a_n/1) = f(a_0, \ldots a_n) = y_j$. If $1 \leq j \leq i$ then $((\phi_i^n)^* f)(P) = \frac{x_{j-1}}{x_i}(P) = \frac{a_{j-1}}{1} = a_{j-1}$. If $i+1 \leq j \leq n$ then $((\phi_i^n)^* f)(P) = \frac{x_j}{x_i}(P) = \frac{a_{j-1}}{1} = a_{j-1}$. Therefore D contains all $y_1, y_2, \ldots y_n$. D also contains all constant functions. Hence D is the whole ring. Therefore $f(\phi_i^n(P)) = ((\phi_i^n)^* f)(P)$. □

Lemma 5.48. *For $g \in K[x_0, \ldots, x_n]_{(x_i)}$, the following are equivalent.*

(i) $g(P) = 0$ *for all* $P \in Y_i = Y \cap U_i^n$.

(ii) $g \in I(Y)_{(x_i)}$

Proof. Assume (i). Put $g = \frac{h}{x_i^m}$ with $h \in K[x_0, \ldots x_n]$ homogeneous of degree m. $g(P) = 0$ for all $P \in Y_i$. Hence $P \in Z(h)$ for all $P \in Y_i$. ($Z(h)$ is the zero set of h). This means $Y_i \subseteq Z(h)$. Therefore $Y \subseteq Z(x_i) \cup Z(h) = Z(x_i h)$. Hence $x_i h \in I(Y)$. Therefore $g = \frac{x_i h}{x_i^{m+1}} \in I(Y)_{(x_i)}$.
Assume (*ii*). Put $g = \frac{h}{x_i^m}$ with $h \in I(Y)$ homogeneous of degree m. Then $Y \subseteq Z(h)$ hence $g(P) = 0$ for all $P \in Y_i$. □

Lemma 5.49. *$A(Y_i) \cong S(Y)_{(\overline{x_i})}$ where $\overline{x_i} = x_i + I(Y)$.*

Proof.

$$(\phi_i^n)^* : K[y_1, \ldots, y_n] \to K[x_0, \ldots, x_n]_{(x_i)}$$
$$y_j \mapsto \begin{cases} \frac{x_{j-1}}{x_i} \text{ if } 1 \leq j \leq i \\ \frac{x_j}{x_i} \text{ if } i+1 \leq j \leq n \end{cases}$$

is a ring homomorphism. Given $T = \{x_i^m, m \geq 0\}$, elements in $K[x_0, \ldots, x_n]_{(x_i)}$ are $\{\frac{f}{b}, f \in K[x_0, \ldots x_n], b \in T, f, b$ homogeneous of same degree$\}$. We must show

$$(\psi_i^n)^* : K[x_0, \ldots, x_n]_{(x_i)} \to K[y_1, \ldots, y_n]$$
$$\frac{g(x_0 \ldots x_n)}{x_i^m} \mapsto g(y_1, \ldots y_i, 1, y_{i+1} \ldots y_n)$$

is a well defined ring homomorphism.
Let $\frac{f(x_0,\ldots,x_n)}{x_i^p} = \frac{g(x_0,\ldots,x_n)}{x_i^q}$ assuming $q \geq p$. Then

$$t(x_i^q f(x_0, \ldots, x_n) - x_i^p g(x_0, \ldots, x_n)) = 0$$

for some $t \in T$.

Hence $x_i^m(x_i^q f(x_0, \ldots, x_n) - x_i^p g(x_0, \ldots, x_n)) = 0$. Since $K[x_0, \ldots, x_n]$ is an integral domain, it contains no zero divisors. This results in

$$x_i^q f(x_0, \ldots, x_n) - x_i^p g(x_0, \ldots, x_n) = 0$$

$$x_i^{q-p} f(x_0, \ldots, x_n) - g(x_0, \ldots, x_n) = 0$$

So we obtain $x_i^{q-p} f(x_0, \ldots, x_n) = g(x_0, \ldots, x_n)$. Substituting in our point gives $1^{q-p} f(y_1, \ldots, y_i, 1, y_{i+1} \ldots, y_n) = g(y_1, \ldots, y_i, 1, y_{i+1} \ldots, y_n)$. Hence $f(y_1, \ldots, y_n) = g(y_1 \ldots, y_n)$. Therefore $(\psi_i^n)^*$ is a well defined map.

Let $\frac{g(x_0,\ldots,x_n)}{x_i^q}, \frac{h(x_0,\ldots,x_n)}{x_i^p} \in K[x_0,\ldots,x_n]_{(x_i)}$.

$$\begin{aligned}(\psi_i^n)^*\big(\frac{g(x_0,\ldots,x_n)}{x_i^q} + \frac{h(x_0,\ldots,x_n)}{x_i^p}\big) &= (\psi_i^n)^*(\frac{x_i^p g(x_0,\ldots,x_n) + x_i^q h(x_0,\ldots,x_n))}{x_i^{p+q}})\\ &= 1^p g(y_1,\ldots y_i, 1, y_{i+1}\ldots,y_n)+\\ &\quad 1^q h(y_1,\ldots,y_i,1,y_{i+1},\ldots,y_n)\\ &= g(y_1,\ldots,y_n) + h(y_1,\ldots,y_n)\\ &= (\psi_i^n)^*\big(\frac{g(x_0,\ldots,x_n)}{x_i^q}\big) + (\psi_i^n)^*\big(\frac{h(x_0,\ldots,x_n)}{x_i^p}\big)\end{aligned}$$

$$\begin{aligned}(\psi_i^n)^*\big(\frac{g(x_0,\ldots,x_n)}{x_i^q}\cdot\frac{h(x_0,\ldots,x_n)}{x_i^p}\big) &= (\psi_i^n)^*(\frac{g(x_0,\ldots,x_n)\cdot h(x_0,\ldots,x_n))}{x_i^{p+q}})\\ &= (\psi_i^n)^*(\frac{gh(x_0,\ldots,x_n)}{x_i^{p+q}})\\ &= gh(y_1,\ldots,y_i,1,y_{i+1},\ldots,y_n)\\ &= g(y_1,1,\ldots,y_n)\cdot h(y_1,1,\ldots,y_n)\\ &= (\psi_i^n)^*\big(\frac{g(x_0,\ldots,x_n}{x_i^q}\big)\cdot(\psi_i^n)^*\big(\frac{h(x_0,\ldots,x_n)}{x_i^p}\big)\end{aligned}$$

$$(\psi_i^n)^*\big(\frac{1}{1}\big) = 1$$

Therefore $(\psi_i^n)^*$ is a well defined ring homomorphism.

Let $\frac{f(x_0,\ldots,x_n)}{x_i^m} \in K[x_0,\ldots,x_n]_{(x_i)}$. Then degree $f(x_0,\ldots,x_n) =$ degree $x_i^m = m$.

$$\begin{aligned}((\phi_i^n)^*\circ(\psi_i^n)^*)\big(\frac{f(x_0,\ldots,x_n)}{x_i^m}\big) &= (\phi_i^n)^*(f(y_1,\ldots y_i,1,y_{i+1},\ldots,y_n))\\ &= f(\frac{x_0}{x_i},\ldots,1,\ldots,\frac{x_n}{x_i})\\ &= f(\frac{x_0}{x_i},\frac{x_1}{x_i},\ldots\frac{x_i}{x_i},\ldots,\frac{x_n}{x_i})\\ &= \frac{f(x_0,\ldots,x_n)}{x_i^m}\end{aligned}$$

Hence $(\phi_i^n)^*\circ(\psi_i^n)^*$ is the identity in $K[x_0,\ldots,x_n]_{(x_i)}$.

Let $g(y_1, \dots, y_n) \in K[y_1, \dots, y_n]$. $D = \{g \in K[y_1, \dots y_n] | ((\psi_i^n)^* \circ (\phi_i^n)^*)(g) = g\} \subseteq K[y_1, \dots y_n]$. $(\psi_i^n)^* \circ (\phi_i^n)^*$ is a ring homomorphism. Hence D is a subring by Lemma 5.46. D contains $y_1, \dots, y_n$, all $c \in K$. Therefore D generates $K[y_1, \dots, y_n]$ as a ring. Hence $D = K[y_1, \dots y_n]$. Therefore $(\psi_i^n)^* \circ (\phi_i^n)^*$ is the identity on $K[y_1, \dots x_n]$.

Hence $K[y_1, \dots, y_n] \cong K[x_0, \dots, x_n]_{(x_i)}$.

We define $I(Y)_{(x_i)} = I(Y)_{x_i} \cap S_{(x_i)}$ in S_x. $(\phi_i^n)^*(I(\phi_i^n(Y_i))) = I(Y)_{(x_i)}$. We see this as follows.

Let $f \in I(\phi_i^n(Y_i))$. Hence $f(\phi_i^n(P)) = 0$ for all $P \in Y_i$. Therefore $((\phi_i^n)^* f)(P) = 0$ for all $P \in Y_i$. Hence $g = (\phi_i^n)^* f \in K[x_0, \dots, x_n]_{(x_i)}$ satisfies $g(P) = 0$ for all $P \in Y_i$. Hence by Lemma 5.48 $g \in I(Y)_{(x_i)}$. Therefore $(\phi_i^n)^*(I(\phi_i^n(Y_i))) \subseteq I(Y)_{(x_i)}$.

Let $g \in I(Y)_{(x_i)}$. By Lemma 5.48 $g(P) = 0$ for all $P \in Y_i$. Since ϕ_i^n is a bijective map $g = (\phi_i^n)^* f$ for a unique $f \in K[y_1, \dots y_n]$. Hence $((\phi_i^n)^* f)(P) = 0$ for all $P \in Y_i$. Therefore $f(\phi_i^n(P)) = 0$ for all $P \in Y_i$. Hence $f \in I(\phi_i^n(Y_i))$. Therefore $I(Y)_{(x_i)} \subseteq (\phi_i^n)^*(I(\phi_i^n(Y_i)))$.
Hence $(\phi_i^n)^*(I(\phi_i^n(Y_i))) = I(Y)_{(x_i)}$. Passing to the quotient as in Lemma 2.85 we obtain $A(Y_i) \cong S(Y)_{(\overline{x_i})}$. □

Lemma 5.50. *Let Y be a projective variety. Let $Y_i = Y \cap U_i^n$ where $U_i^n \subseteq \mathbf{P}^n$ is the open set defined by $x_i \neq 0$. Then $K(Y) \cong K(Y_i)$.*

Proof. Let $x \in K(Y)$. Then $x = (U, f)$ where U is an open set of Y and f is regular on U. We define

$$\begin{aligned} \tau : & K(Y) \to K(Y_i) \\ & (U, f) \mapsto (U \cap Y_i, f_{|U \cap Y_i}) \end{aligned}$$

Y is irreducible therefore $Y \supseteq Y \cap U_i^n \neq \emptyset$. $U \subseteq Y$ is non-empty and open. Hence we have $Y \cap U_i^n \cap U \neq \emptyset$. Therefore $Y_i \neq \emptyset$. By the subspace topology on $Y_i, U \cap Y_i$ is an open subset of Y_i. By Lemma 5.15 $f_{|U \cap Y_i}$ is regular.
Let $(U, f) \sim (V, g)$. Then $f = g$ on $U \cap V$. i.e $f(P) = g(P)$ for all $P \in U \cap V$. $\tau((U, f)) = (U \cap Y_i, f_{|U \cap Y_i})$. $\tau((V, g)) = (V \cap Y_i, g_{|V \cap Y_i})$. Let $P \in U \cap V \cap Y_i$. We have $f(P) = g(P)$ on $U \cap V$. Hence $f_{|U \cap Y_i}(P) = g_{|V \cap Y_i}(P)$ on $U \cap V \cap Y_i$. Therefore $(U \cap Y_i, f_{|U \cap Y_i}) \sim (V \cap Y_i, g_{|V \cap Y_i})$ and τ is well defined.

Let $x, y \in K(Y)$. Then $x = (U, f)$ where U is an open set of Y and f is regular on U and $y = (V, g)$ where V is an open set of Y and g is regular on V.

$$\begin{aligned}\tau\big((U,f)+(V,g)\big) &= \tau(U\cap V, f+g)\\ &= (U\cap V\cap Y_i, (f+g)_{|U\cap V\cap Y_i})\\ &= (U\cap Y_i\cap V\cap Y_i, (f_{|U\cap V\cap Y_i} + g_{|U\cap V\cap Y_i})\\ &= (U\cap Y_i, f_{|U\cap Y_i}) + (V\cap Y_i, g_{|V\cap Y_i})\\ &= \tau\big((U,f)+\tau(V,g)\big)\end{aligned}$$

$$\begin{aligned}\tau\big((U,f)(V,g)\big) &= \tau(U\cap V, fg)\\ &= (U\cap V\cap Y_i, (fg)_{|U\cap V\cap Y_i})\\ &= (U\cap Y_i\cap V\cap Y_i, (f_{|U\cap V\cap Y_i} g_{|U\cap V\cap Y_i})\\ &= (U\cap Y_i, f_{|U\cap Y_i})(V\cap Y_i, g_{|V\cap Y_i})\\ &= \tau\big((U,f)\tau(V,g)\big)\end{aligned}$$

$$\begin{aligned}\tau\big((Y,1)\big) &= \tau(Y\cap Y_i, 1_{|Y\cap Y_i})\\ &= (Y_i, 1)\end{aligned}$$

Hence τ is a ring homomorphism.
We define

$$\begin{aligned}\tau' :&K(Y_i) \to K(Y)\\ &(U,f) \mapsto (U,f)\end{aligned}$$

τ' is clearly well defined.

Let $x \in K(Y_i)$. Then $x = (U, f)$.

$$\begin{aligned}(\tau\circ\tau')(U,f) &= \tau(\tau'(U,f))\\ &= \tau(U,f)\\ &= (U\cap Y_i, f_{|U\cap Y_i})\\ &= (U,f)\end{aligned}$$

Let $x \in K(Y)$. Then $x = (U, f)$.

$$\begin{aligned}(\tau' \circ \tau)(U,f) &= \tau'(\tau(U,f))\\ &= \tau'(U \cap Y_i, f_{|U\cap Y_i})\\ &= (U,f)\end{aligned}$$

Therefore τ is a ring isomorphism and $K(Y) \cong K(Y_i)$ □

By Lemma 5.43 part d, $K(Y_i) \cong Q(A(Y_i))$ where $Q(A(Y_i))$ is the quotient field of $A(Y_i)$. Next we show $Q(A(Y_i)) \cong Q(S(Y)_{(x_i)}) \cong S(Y)_{((0))}$.

Corollary 5.51. $Q(A(Y_i)) \cong Q(S(Y)_{(\overline{x_i})})$.

Proof. By Lemma 5.49 $A(Y_i) \cong S(Y)_{(\overline{x_i})}$. Therefore using Lemma 4.22 we obtain $Q(A(Y_i)) \cong Q(S(Y)_{(\overline{x_i})})$. □

Lemma 5.52. $Q(S(Y)_{(\overline{x_i})}) \cong S(Y)_{((0))}$

Proof. $Q(S(Y)_{(\overline{x_i})}) = \{\frac{f}{g} : f,g \in S(Y)_{(\overline{x_i})}, g \neq 0\}$. Another way of interpreting this set is as a form of localization, namely $(S(Y)_{(\overline{x_i})})_{(0)}$. Hence we will prove $(S(Y)_{(\overline{x_i})})_{(0)} \cong S(Y)_{((0))}$. By Lemma 4.19 we have that $R_{(\mathfrak{p})} \cong (R_{(x)})_{\mathfrak{p}_{(x)}}$. Here $R = S(Y), (0) = \mathfrak{p}$ and $x = \overline{x_i}$. $(0)_{(\overline{x_i})} \subseteq S(Y)_{(\overline{x_i})}$. $\mathfrak{p}_{(x)} = (0)_{(\overline{x_i})}$=$\{\frac{0}{\overline{x_i}^r} | 0 \in (0), r \geq 0\} = \{0\} \subseteq S(Y)_{(\overline{x_i})}$. Hence by Lemma 4.19 $(S(Y)_{(\overline{x_i})})_{(0)} \cong S(Y)_{((0))}$. □

Lemma 5.53. *Let Y be a projective variety. Then $K(Y) \cong S(Y)_{((0))}$.*

Proof.

$$\begin{aligned} K(Y) &\cong K(Y_i) \text{ by Lemma 5.50}\\ K(Y_i) &\cong Q(A(Y_i)) \text{ by Theorem 5.43}\\ Q(A(Y_i)) &\cong Q(S(Y)_{(\overline{x_i})}) \text{ by Corollary 5.51}\\ Q(S(Y)_{(\overline{x_i})}) &\cong S(Y)_{((0))} \text{ by Lemma 5.52}\end{aligned}$$

Therefore $K(Y) \cong S(Y)_{((0))}$. □

We will now show that the only regular functions on $\mathcal{O}(Y)$ are the constant functions. To do this we briefly discuss modules and submodules.

5.4.1 Modules

Definition 5.54. Let R be a ring, $S \subseteq R$ a subset and $f \in R$. Then $S \cdot f = \{s \cdot f | s \in S\} \subseteq R$

Lemma 5.55. *Let A be a ring, $B \subseteq A$ a subring and $f \in A$. Then,*

$$B[f] := \{b_0 + b_1 f + b_2 f^2 + \cdots + b_n f^n : n \geq 0, b_0, b_1, \ldots, b_n \in B\} \subseteq A$$

is a subring of A.

Proof. It is clear $B[f] \subseteq A$. To prove $B[f]$ is a subring of A we must show the following conditions hold.

(i) $B[f]$ is closed under addition.

(ii) $B[f]$ is closed under multiplication.

(iii) $1_A \in B[f]$.

(iv) for $x \in B[f], -x \in B[f]$.

Proof of (i).
Let $b_0 + b_1 f + b_2 f^2 + \cdots + b_n f^n, c_0 + c_1 f + c_2 f^2 + \cdots + c_m f^m \in B[f]$. WLOG we can assume $n \geq m$. Then,

$$\begin{aligned} b_0 + b_1 f + b_2 f^2 + \cdots + b_n f^n + c_0 + c_1 f + c_2 f^2 + \cdots + c_m f^m = \\ (b_0 + c_0) + (b_1 + c_1) f + (b_2 + c_2) f^2 + \cdots + (b_m + c_m) f^m \\ + b_{m+1} f^{m+1} + b_{m+2} f^{m+2} + \cdots + b_n f^n \end{aligned}$$

Since $b_0 + c_0, b_1 + c_1, b_2 + c_2, \ldots, b_m + c_m, b_{m+1}, b_{m+2}, \ldots b_n$ are in B, this sum is in $B[f]$.
Proof of (ii)
Let $b_0 + b_1 f + b_2 f^2 + \cdots + b_n f^n, c_0 + c_1 f + c_2 f^2 + \cdots + c_m f^m \in B[f]$. Then,

$$\begin{aligned} (c_0 + c_1 f + c_2 f^2 + \cdots + c_m f^m)(b_0 + b_1 f + b_2 f^2 + \cdots + b_n f^n) = \\ (\sum_{i=0}^{m} c_i f^i)(\sum_{j=0}^{n} b_j f^j) = \\ \sum_{k=o}^{m+n} (\sum_{i+j=k} c_i b_j) f^k \end{aligned}$$

Since the sum of the $c_i b_j \in B$, this sum is in $B[f]$.
Proof of (iii). Since B is a subring $1_A \in B$. Letting $b_0 = 1_A$ we see $1_A \in B[f]$.

Proof of (iv). Let $b_0 + b_1 f + b_2 f^2 + \cdots + b_n f^n \in B[f]$. Since B is a subring, for each $b_i \in B$ there is an inverse $-b_i \in B$. Hence $-b_0 - b_1 f - b_2 f^2 - \cdots - b_n f^n \in B[f]$. But $-b_0 - b_1 f - b_2 f^2 - \cdots - b_n f^n = -(b_0 + b_1 f + b_2 f^2 + \cdots + b_n f^n)$. Therefore $-(b_0 + b_1 f + b_2 f^2 + \cdots + b_n f^n) \in B[f]$. □

Lemma 5.56. *Let R be a commutative ring, $S \subseteq R$ a subring. R is an S-module.*

Proof. R is an additive group. We define a map

$$S \times R \to R$$

$$(s, r) \mapsto sr$$

where $s \in S, r \in R$ and sr is the multiplication in R.
Let e be the unit element in S and $r \in R$. By definition of S being a subring of R, e is the unit element in R. By definition of R being a ring $er = r$ for all $r \in R$.
Let $s \in S, r_1, r_2 \in R$. Since $S \subseteq R$, $s \in R$. Hence $s(r_1 + r_2) = sr_1 + sr_2$ in R.
Let $s_1, s_2 \in S, r \in R$. Since $S \subseteq R$, $s_1, s_2 \in R$. Hence $(s_1 + s_2)r = s_1 r + s_2 r$ in R.
Let $s_1, s_2 \in S, r \in R$. Since $S \subseteq R$, $s_1, s_2 \in R$. Hence $(s_1 s_2)r = s_1(s_2 r)$ in R.
Therefore R is an S-module. □

Definition 5.57. Let R be a ring and M an R-module. Let N be a subgroup of M. We call N a submodule of M, if whenever $n \in N$ and $r \in R$, then $nr \in N$.

Definition 5.58. Let R be a ring and M an R-module. M is noetherian as an R-module if every submodule is finitely generated.

Lemma 5.59. *Let R be a ring and M an R-module. Let N be a submodule of M. N is an R module.*

Proof. If N is a subgroup of M, then N is an additive group. Since M is an R-module we have a map

$$R \times M \to M$$

$$(r, m) \mapsto rm$$

with $m \in M, r \in R$ and rm is the multiplication in M.We need to check when given (r, n) for $n \in N, r \in R$ that rn is in N. Given that N is a submodule of M, then by definition rn is in N.
Hence we have a multiplication map given by:

$$R \times N \to N$$

$$(r, n) \mapsto rn$$

for $n \in N, r \in R, rn \in N$

Let e be the unit element in R and $n \in N$. Given that N is an submodule, $en = n$.
Let $r \in R, n_1, n_2 \in N$. Since $N \subseteq M$, $n_1, n_2 \in M$. Hence $r(n_1 +_M n_2) = rn_1 + rn_2$ where $+_M$ is addition in M.
Let $r_1, r_2 \in R, n \in N$. Since $N \subseteq M$, $n \in M$. Hence $(r_1 +_M r_2)n = r_1 n + r_2 n$.
Let $r_1, r_2 \in R, n \in N$. Since $N \subseteq M$, $n \in M$. Hence $(r_1 r_2)n = r_1(r_2 n)$.
Therefore N is an R-module. □

Lemma 5.60. *Let M be an R-module. Let $N_1, N_2 \subseteq M$ be R-submodules of M with $N_1 \subseteq N_2$. Then N_1 is an R-submodule of N_2.*

Proof. Given that M is an R-module and N_1, N_2 are R-submodules of M then by definition 5.57, N_1, N_2 are subgroups of M. For $N_1 \subseteq N_2$ then N_1 is a subgroup of N_2. By definition of submodule, for $n_1 \in N_1, r \in R$ then $n_1 r \in N_1$. Therefore N_1 is an R-submodule of N_2. □

Definition 5.61. Let R be a ring. Let M be an R -module. A module endomorphism of M is a module homomorphism from M to itself.

Definition 5.62. Let R be a ring, R' a subring of R. An element x of R is said to be integral over R' if x is a root of a monic polynomial with coefficients in R', that is if x satisfies an equation of the form

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0$$

where the a_i are elements of R'.

Lemma 5.63. *Let R be a ring, R' a subring of R and $x \in R$. The following statements are equivalent:*

(i) x is integral over R'.

(ii) $R'[x]$ is a finitely generated R'-module.

Proof. Assume (i). Let $x \in R$ be integral over R'. Then, x is a root of a monic polynomial with coefficients in R'. Hence x satisfies an equation of the form

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0$$

where the a_i are elements of R'. Hence

$$x^n = -a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_n$$

$$x^n = -(a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n)$$

Let S' be the submodule of $R'[x]$ generated by $\{1, x, x^2, \ldots x^{n-1}\}$. For all s with $0 \leq s \leq n-1$, it is clear $x^s \in S'$. We want to show $x^n, x^{n+1}, x^{n+2}, \ldots x^{n+r} \in S'$ for all $r \geq 0$. We have

$$x^n = -(a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n)$$

We see x^n is a linear combination of elements already in S'. Hence $x^n \in S'$.

$$x^{n+1} = -(a_1 x^n + a_2 x^{n-1} + a_3 x^{n-2} \cdots + a_n x)$$

We see x^{n+1} is a linear combination of elements already in S' including x^n. Hence $x^{n+1} \in S'$.

$$x^{n+2} = -(a_1 x^{n+1} + a_2 x^n + a_3 x^{n-1} \cdots + a_n x^2)$$

We see x^{n+2} is a linear combination of elements already in S' including x^{n+1}. Hence $x^{n+2} \in S'$. Using induction we make the claim $x^{n+r} \in S'$ for $r \geq 0$. Assume $x^{n+r-1}, x^{n+r-2}, \ldots x^r \in S'$.

$$x^{n+r} = -(a_1 x^{n+r-1} + a_2 x^{n+r-2} + a_3 x^{n+r-3} \cdots + a_n x^r)$$

Hence from our assumption we see $x^{n+r} \in S'$ as it is a linear combination of elements already in S'. It is obvious $S' \subseteq R'[x]$. Let $a \in R'[x]$. Then $a = r'_0 + r'_1 x + r'_2 x^2 + \cdots + r'_t x^t$ for some $t \geq 0, r'_0, r'_1, r'_2, \ldots r'_t \in R'$. It is clear from our induction proof that $\sum r'_i x^i$ for $0 \leq i \leq t$ is in S'. Therefore $R'[x] \subseteq S'$. Hence $S' = R'[x]$. Therefore (i) implies (ii).

Assume (ii). We are given $R'[x]$ is a finitely generated R'-module. Let $\{x_1, x_2, \ldots x_n\}$ be a set of generators of $R'[x]$ as an R'-module. We will show that x satisfies an equation of the form

$$x^n + r'_1 x^{n-1} + r'_2 x^{n-2} + \cdots + r'_n = 0$$

for some $r'_1, r'_2 \ldots, r'_n \in R'$. We have $x x_i \in R'[x]$. As $\{x_1, x_2, \ldots x_n\}$ are a set of generators of $R'[x]$, we have that $x x_i = \sum_{j=1}^{n} r'_{ij} x_j, \quad 1 \leq i \leq n$ for some $r'_{ij} \in R'$. Hence $x x_i - \sum_{j=1}^{n} r'_{ij} x_j = 0$. Using the Kronecker Delta which tells us

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

we can write $x x_i - \sum_{j=1}^{n} r'_{ij} x_j = 0$ as $\sum_{j=1}^{n} \delta_{ij} x x_j - \sum_{j=1}^{n} r'_{ij} x_j = 0$. Hence we obtain

$$\sum_{j=1}^{n} (\delta_{ij} x - r'_{ij}) x_j = 0$$

$$\begin{aligned}
\text{for } i = 1 \text{ we have, } (x - r'_{11})x_1 - r'_{12}x_2 + \cdots - r'_{1n}x_n &= 0 \\
\text{for } i = 2 \text{ we have, } -r'_{21}x_1 + (x - r'_{22})x_2 + \cdots - r'_{2n}x_n &= 0 \\
\text{for } i = 3 \text{ we have, } -r'_{31}x_1 + -r'_{32}x_2 + \cdots - r'_{3n}x_n &= 0 \\
\vdots \quad & \quad \vdots \\
\text{for } i = n \text{ we have, } -r'_{n1}x_1 - r'_{n2}x_2 - \cdots + (x - r'_{nn})x_n &= 0
\end{aligned}$$

In matrix form $A \times B = C$, this set of equations become,

$$\begin{pmatrix} x - r'_{11} & -r'_{12} & -r_{13} & \cdots & -r'_{1n} \\ -r'_{21} & x - r'_{22} & -r_{23} & \cdots & -r'_{2n} \\ -r'_{31} & -r'_{32} & x - r'_{33} & \cdots & -r'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r'_{n1} & -r'_{n2} & -r'_{n3} & \cdots & x - r'_{nn} \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} x - r'_{11} & -r'_{12} & -r_{13} & \cdots & -r'_{1n} \\ -r'_{21} & x - r'_{22} & -r_{23} & \cdots & -r'_{2n} \\ -r'_{31} & -r'_{32} & x - r'_{33} & \cdots & -r'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r'_{n1} & -r'_{n2} & -r'_{n3} & \cdots & x - r'_{nn} \end{pmatrix}$$

Multiplication of an $n \times n$ matrix A by its adjoint is equal to the det$A$ multiplied by I where I is the identity matrix. i.e $Adj(A) \cdot A = det(A) \cdot (I)$. So multiplying our set of equations in matrix form by the adj A we get the result,

$$\begin{pmatrix} D & 0 & 0 & 0 & \cdots & 0 \\ 0 & D & 0 & 0 & \cdots & 0 \\ 0 & 0 & D & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & D \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $D = \det(A)$.
Hence we have

$$\begin{aligned} Dx_1 &= 0 \\ Dx_2 &= 0 \\ Dx_3 &= 0 \\ &\vdots \\ Dx_n &= 0 \end{aligned}$$

Hence D annihilates each x_i. Multiplying each x_i by any $a_i \in R'$ we have

$$\begin{aligned} Da_1x_1 &= 0 \\ Da_2x_2 &= 0 \\ Da_3x_3 &= 0 \\ &\vdots \\ Da_nx_n &= 0 \end{aligned}$$

$Da_1x_1 + Da_2x_2 + Da_3x_3 + \cdots + Da_nx_n = D(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = 0$. Hence $D\sum_1^n a_ix_i = 0$. Therefore D annihilates any linear combination. Choosing a linear combination in $R'[x]$ in particular 1, we have $D \cdot 1 = 0$. Hence $D = 0$.

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$$

where $\sigma : \{1 \dots n\} \mapsto \{1 \dots n\}$, $\text{sgn}(\sigma) = \pm 1$ and S_n is the symmetric group on n. Each $a_{b\sigma(b)}$ for $1 \leq b \leq n$ either gives $x - r'_{b\sigma(b)}$ or $-r'_{b\sigma(b)}$. This results in a polynomial of degree at most n. Each term of the polynomial is associated with a permutation. Let $\sigma = Id$ permutation. Then we have

$$\text{sgn}(Id)a_{1Id(1)}a_{2Id(2)} \dots a_{nId(n)} = a_{11}a_{22}a_{33} \dots a_{nn}$$

Hence this permutation gives the term $(x - r'_{11})(x - r'_{22}) \dots (x - r'_{nn})$. This term is monic of degree n. All other terms in the polynomial will have degree less than n. This is because at least one $a_{b\sigma(b)}$ will not be $x - r'_{b\sigma(b)}$. The sum of all these terms is 0. Hence we have a monic polynomial of the required form. Therefore (ii) implies (i).

Hence (i) and (ii) are equivalent statements. □

Theorem 5.64. *Let Y be a projective variety in $\mathbf{P}^n$. $\mathcal{O}(Y) = K$. In other words the only regular functions on Y are the constant functions.*

Proof. Let $f \in \mathcal{O}(Y)$ be a global regular function. Then for each i, f is regular on Y_i. By Lemma 5.43 (a) $f \in A(Y_i)$. Since $A(Y_i) \cong S(Y)_{(x_i)}$ we can express f as $\frac{g_i}{x_i^{N_i}}$ with $g_i \in S(Y)$, homogeneous of degree N_i. It is clear $S(Y) \subset Q(S(Y))$ is a subring. We can also view $\mathcal{O}(Y)$ and $K(Y)$ as subrings of $Q(S(Y))$ since $\mathcal{O}(Y) \subseteq K(Y) \cong S(Y)_{((0))} \subseteq Q(S(Y))$. Hence $x_i^{N_i} f \in S(Y)_{N_i}$ is homogeneous of degree N_i. Choose N such that $N \geq \sum N_i$. This means $N \geq N_0 + N_1 + \dots + N_n$. Then each element in $S(Y)_N$ can be expressed as a linear combination of monomials of degree N in $x_0, \dots x_n$. Hence $S(Y)_N$ is spanned as a K vector space by these monomials. In any such monomial of degree N in $x_0, \dots x_n$ at least one x_i has a power $\geq N_i$. We show this as follows.
Let $x_0^{M_0} x_1^{M_1} \dots x_n^{M_n}$ be a monomial of degree N. Then $M_0 + M_1 + \dots + M_n = N \geq N_0 + N_1 + \dots + N_n$. Assume no x_i occurs to a power $\geq N_i$. This means $M_i \ngeq N_i$ or in other words $M_i < N_i$ for each i with $0 \leq i \leq n$. This would mean $\sum M_i < \sum N_i$. But $\sum M_i = N$ and $N \geq N_0 + N_1 + \dots + N_n$. Hence we have a contradiction. Therefore in any such monomial of degree N in $x_0, \dots x_n$ at least one x_i has a power $\geq N_i$. From this theory we can conclude that $S(Y)_N \cdot f \subset S(Y)_N$ where $S(Y)_N \cdot f = \{a \cdot f | a \in S(Y)_N, \}$. We see this as follows.By definition 5.54 $S(Y) \cdot f \subseteq Q(S(Y))$. Let $x_0^{M_0} x_1^{M_1} \dots x_n^{M_n} \in S(Y)_N$ be homogeneous of degree N. $M_0 + M_1 + \dots + M_n = N$. We choose i such that $M_i \geq N_i$. We can express $x_0^{M_0} x_1^{M_1} \dots x_n^{M_n} \cdot f$ as $x_0^{M_0} x_1^{M_1} \dots x_n^{M_n} \cdot \frac{g_i}{x_i^{N_i}}$ where $g_i \in S(Y)$ is of degree N_i.
Then,

$$x_0^{M_0} x_1^{M_1} \dots x_i^{M_i} \cdots x_n^{M_n} \cdot f =$$
$$x_0^{M_0} x_1^{M_1} \dots x_i^{M_i} \cdots x_n^{M_n} \cdot \frac{g_i}{x_i^{N_i}} =$$
$$x_0^{M_0} x_1^{M_1} \dots x_i^{M_i - N_i} \cdots x_n^{M_n} \cdot g_i.$$

The sum of the degrees gives $M_0 + M_1 + \dots + M_i - N_i + \dots + M_n + N_i = N$. Hence $x_0^{M_0} x_1^{M_1} \dots x_i^{M_i - N_i} \cdots x_n^{M_n} \cdot g_i$ is homogeneous of degree N Therefore

$x_0^{M_0}x_1^{M_1}\dots x_n^{M_n}\cdot f \in S(Y)_N$. Multiplication by a scalar will also be in $S(Y)_N$. i.e $ax_0^{M_0}x_1^{M_1}x_2^{M_2}\dots x_i^{M_i}\dots x_n^{M_n}\cdot f \in S(Y)_N$. The sum of finitely many elements is in $S(Y)_N$. Therefore $S(Y)_N \cdot f \subset S(Y)_N$. By iteration we see,

$$\begin{aligned} S(Y)_N \cdot f \subset& S(Y)_N \\ S(Y)_N \cdot f \cdot f \subset& S(Y)_N \cdot f \subseteq S(Y)_N \\ S(Y)_N \cdot f^2 \cdot f \subset& S(Y)_N \cdot f \subseteq S(Y)_N \\ S(Y)_N \cdot f^3 \cdot f \subset& S(Y)_N \cdot f \subseteq S(Y)_N \\ &. \\ &. \\ &. \\ S(Y)_N \cdot f^q \subseteq& S(Y)_N \end{aligned}$$

for all $q > 0$. In particular $x_0^N f^q \in S(Y)_N$ and $S(Y)_N \subseteq S(Y)$. Therefore $x_0^N f^q \in S(Y)$ for all $q > 0$. By Lemma 5.55 $S(Y)[f]$ is a subring of $Q(S(Y))$. By definition $S(Y)[f] = \{b_0+b_1f+b_2f^2+\cdots b_tf^t|b_0, b_1, b_2, \dots b_t \in S(Y)\}$. Since $x_0^N f^q \in S(Y)$ for all $q > 0$, $f^q \in x_0^{-N}S(Y)$ for all $q > 0$. Let $b_0 + b_1f + b_2f^2 + \cdots b_tf^t \in S(Y)[f]$. For all i with $0 \leq i \leq t$ we have $b_i \in S(Y)$and $f^i \in x_0^{-N}S(Y)$. Hence $b_if^i \in x_0^{-N}S(Y)$ and thus $b_if^i = x_0^{-N}a_i$ for some $a_i \in S(Y)$. Therefore,

$$\begin{aligned} \sum_{i=0}^{t} b_if^i &= \sum_{i=0}^{t} x_0^{-N}a_i \\ &= x_0^{-N}\sum_{i=0}^{t} a_i \end{aligned}$$

$\sum_{i=0}^{t} a_i \in S(Y)$. Therefore $x_0^{-N}\sum_{i=0}^{t} a_i \in x_0^{-N}S(Y)$. Hence the subring $S(Y)[f]$ is contained in $x_0^{-N}S(Y)$.
By Lemma 5.56 $Q(S(Y))$ is an $S(Y)$ module. $x_0^{-N}S(Y)$ is a subgroup of $Q(S(Y))$. Let $x_0^{-N}\dot{g} \in x_0^{-N}S(Y)$ and $h \in S(Y)$. Then $x_0^{-N}\cdot g \cdot h = x_0^{-N}(gh)$. $hg \in S(Y)$. Hence $x_0^{-N}(hg) \in x_0^{-N}S(Y)$. Therefore $x_0^{-N}S(Y)$ is a submodule of $Q(S(Y))$. By Lemma 5.59 $x_0^{-N}S(Y)$ is an $S(Y)$ module. Let $x \in x_0^{-N}S(Y)$. Then $x = x_0^{-N}j$ for some $j \in S(Y)$. Hence $\{x_0^{-N}\}$ is a generating set for $x_0^{-N}S(Y)$. Therefore $x_0^{-N}S(Y)$ is a finitely generated $S(Y)$ module. By Lemma 5.60 $S(Y)[f]$ is an $S(Y)$-submodule of $x_0^{-N}S(Y)$. By Lemma 2.5 $S = K[x_0, x_1, \cdots, x_n]$ is a noetherian ring. By Lemma 2.84 $S/I(Y)$ is noetherian. Therefore $S(Y)$ is noetherian. $x_0^{-N}S(Y)$ is noetherian as an $S(Y)$-module.[SLA, Prop 1.4 page 415]. By definition every sub-module of $x_0^{-N}S(Y)$ is a finitely generated sub-module. Hence $S(Y)[f]$ is a finitely generated $S(Y)$-module. By Lemma 5.63 f is integral over $S(Y)$. This means f is a root of a monic polynomial with coefficients in $S(Y)$. i.e there exists

$a_1, a_2, \ldots, a_m \in S(Y)$ such that

$$f^m + a_1 f^{m-1} + a_2 f^{m-2} + \cdots + a_m = 0 \tag{3}$$

$f = \frac{g}{x_0^N}$ with $g \in S(Y)$ of degree N. We replace our f in equation (3) and obtain

$$\frac{g^m}{x_0^{Nm}} + a_1 \frac{g^{m-1}}{x_0^{N_{m-1}}} + a_2 \frac{g^{m-2}}{x_0^{N_{m-2}}} + \cdots + a_m = 0 \tag{4}$$

Multiplying equation (4) by x_0^{Nm} we obtain (in $S(Y)$)

$$g^m + a_1 g^{m-1} x_0^N + a_2 g^{m-2} x_0^{2N} + \cdots + a_m x_0^{Nm} = 0$$

Here $g^m, g^{m-1}x_0^N, g^{m-2}x_0^{2N}, \ldots, x_0^{Nm}$ are homogeneous of degree Nm. This means each term of the equation is of degree $\geq Nm$. Since $S(Y)$ is graded, we can replace the a_i by their homogeneous components of degree 0 and obtain (in $S(Y)$)

$$g^m + a_{1,0} g^{m-1} x_0^N + a_{2,0} g^{m-2} x_0^{2N} + \cdots + a_{m,0} x_0 = 0$$

Hence (in $Q(S(Y))$)

$$f^m + a_{1,0} f^{m-1} + a_{2,0} f^{m-2} + \cdots + a_{m,0} = 0$$

Therefore f is a root of some polynomial $p(t) = t^m + a_{1,0}t^{m-1} + a_{2,0}t^{m-2} + \cdots + a_{m,0} \in k[t]$. As K is algebraically closed we have

$$p(t) = (t - \alpha_1) \ldots (t - \alpha_n)$$

for $\alpha_1, \alpha_2 \ldots \alpha_n \in k$. But $p(f) = 0$, hence

$$(f - \alpha_1) \ldots (f - \alpha_n) = 0$$

Therefore $f - \alpha_i = 0$ for some $i \in \{1, 2 \ldots n\}$. Hence $f = \alpha_i$. Therefore $f \in K$. This shows the only regular functions are constant. □

We will look at some examples.

Lemma 5.65. *Let f_0 and f_1 be polynomials in one variable over K. Assume*

$$f_0(t) = f_1(1/t) \text{ for all } t \neq 0.$$

Then f_0 and f_1 are constant functions. Moreover f_0 and f_1 are the same constant function.

Proof.

$$f_0(t) = b_m t^m + b_{m-1} t^{m-1} + \cdots + b_0 \text{ with } b_m, b_{m-1}, \ldots, b_0 \in K$$

$$f_1(s) = c_l s^l + c_{l-1} s^{l-1} + \cdots + c_0 \text{ with } c_l, c_{l-1}, \ldots, c_0 \in K$$

$$f_1(\frac{1}{t}) = \frac{c_l}{t^l} + \frac{c_{l-1}}{t^{l-1}} + \cdots + c_0 \text{ with } c_l, c_{l-1}, \ldots, c_0 \in K$$

Given that

$$f_0(t) = f_1(1/t)$$

for all $t \neq 0$. we obtain

$$b_m t^m + b_{m-1} t^{m-1} + \cdots + b_0 = \frac{c_l}{t^l} + \frac{c_{l-1}}{t^{l-1}} + \ldots c_0$$

We multiply across by t^l and see that

$$b_m t^{m+l} + b_{m-1} t^{m+l-1} + b_{m-2} t^{m+l-2} + \cdots + b_0 t^l = c_l + c_{l-1} t + c_{l-2} t^2 + \ldots c_0 t^l$$

Therefore

$$b_m t^{m+l} + b_{m-1} t^{m+l-1} + b_{m-2} t^{m+l-2} + \cdots + b_0 t^l - c_l - c_{l-1} t - c_{l-2} t^2 - \cdots - c_0 t^l = 0$$

$$b_m t^{m+l} + b_{m-1} t^{m+l-1} + \cdots + b_1 t^{l+1} + (b_0 - c_0) t^l - c_1 t^{l-1} - c_2 t^{l-2} - \cdots - c_l = 0$$

for all $t \neq 0$. K is an algebraically closed field. Therefore by Lemma 2.2 K is infinite. Hence we have infinitely many roots. Therefore it must be the zero polynomial. ie $b_m, b_{m-1}, \ldots b_1, -c_1 \cdots - c_l$ are all zero. Hence

$$f_0(t) = b_0$$

and

$$f_1(1/t) = c_0$$

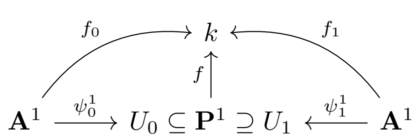
Also $b_0 - c_0 = 0$ from our equation. Hence $b_0 = c_0$. Therefore $f_0(t)$ and $f_1(1/t)$ are the same constant function b_0. □

Example 5.66. Let $Y = \mathbf{P}^1$. $\mathbf{P}^1 = \{[x_0 : x_1]\}$. Let $f \in \mathcal{O}(\mathbf{P}^1)$ be a global regular function. Then

$$f_0 := (\psi_0^1)^*(f|_{U_0}) \in K[y_1]$$

$$f_1 := (\psi_1^1)^*(f|_{U_1}) \in K[y_0]$$

(see diagram).



Let $Q \in U_0 \cap U_1$ i.e $Q = [1 : t], t \neq 0$. Then $Q = \psi_0^1(t) = \psi_1^1(1/t)$. Hence $f(Q) = f(\psi_0^1(t)) = f_0(t)$. Also $f(Q) = f(\psi_1^1(1/t)) = f_1(1/t)$. Hence $f_0(t) = f_1(1/t)$ for all $t \neq 0$.

By Lemma 5.65, f_0 and f_1 are the same constant function b_0. If $P \in U_0$, then $P = \psi_0^1(t)$ for $t \in \mathbf{A}^1$. Hence $f(P) = f(\psi_0^1(t)) = f_0(t) = b_0$. If $P \in U_1$, then $P = \psi_1^1(s)$ for $s \in \mathbf{A}^1$. Hence $f(P) = f(\psi_1^1(s)) = f_1(s) = b_0$.
Hence f is constant on all of $\mathbf{P}^1$ as required.

Before we look at the example of $\mathbf{P}^2$ we first prove the following lemma.

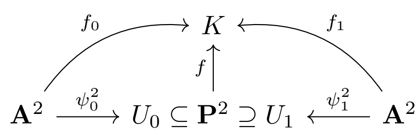
Lemma 5.67. *Let $g(x,y) \in K[x,y]$ be a polynomial in two variables over K such that $g(x,y) = 0$ for all $x \neq 0$ and all y. Then g is the zero polynomial.*

Proof. Viewing g as a function on $\mathbf{A}^2$, $g(P) = 0$ for all $P \neq (0,s), P \in \mathbf{A}^2$. i.e $P \in Z(f)$ for all $P \neq (0,s)$. In other words $\{\mathbf{A}^2 \backslash \{(0,s), s \in k\} \subseteq Z(f)$. From the Zariski topology on $\mathbf{A}^n$ we have $Z(g)$ is closed. $\{(0,s), s \in k\}$ is a closed subset of $\mathbf{A}^2$. Hence $\{\mathbf{A}^2 \backslash \{(0,s), s \in K\}$ is open. Therefore we have an open, non-empty subset of an irreducible space. Hence by Theorem 1.46 $\{\mathbf{A}^2 \backslash \{(0,s), s \in K\}$ is irreducible and dense. By Lemma 1.13 the only closed subset of $\mathbf{A}^2$ that contains $\{\mathbf{A}^2 \backslash \{(0,s), s \in k\}$ is $\mathbf{A}^2$ itself. Hence $Z(g) = \mathbf{A}^2$. Therefore $g \in I(\mathbf{A}^2) = (0)$. Hence $g = 0$. □

Example 5.68. Let $Y = \mathbf{P}^2$. $\mathbf{P}^2 = \{[x_0 : x_1 : x_2]\}$. Let $f \in \mathcal{O}(\mathbf{P}^2)$ be a global regular function. Then

$$f_0 := (\psi_0^2)^*(f|_{U_0})$$
$$f_1 := (\psi_1^2)^*(f|_{U_1})$$
$$f_2 := (\psi_2^2)^*(f|_{U_2})$$

(see diagram).



Let $P \in U_0 \cap U_1$, i.e $P = [1 : t : s], t \neq 0$. Then $P = \psi_0^2(t,s) = \psi_1^2(1/t, s/t)$. Hence $f(P) = f(\psi_0^2(t,s)) = f_0(t,s)$. Also $f(P) = f(\psi_1^2(1/t, s/t)) = f_1(1/t, s/t)$. Hence $f_0(t,s) = f_1(1/t, s/t)$ for all $t \neq 0$.

$f_0 \in K[y_1, y_2]$ hence we can express it as $f_0(y_1,y_2) = \sum_{i=0}^m \sum_{j=0}^n a_{ij} y_1^i y_2^j$ with $a_{ij} \in K$. Therefore

$$f_0(t,s) = \sum_{i=0}^m \sum_{j=0}^n a_{ij} t^i s^j$$

$f_1 \in K[y_0, y_2]$ hence we can express it as $f_1(y_0,y_2) = \sum_{k=0}^p \sum_{l=0}^r c_{kl} y_0^k y_2^l$ with $c_{kl} \in K$. Therefore

$$f_1(\frac{1}{t}, \frac{s}{t}) = \sum_{k=0}^p \sum_{l=0}^r c_{kl} \cdot \frac{1}{t^k} \cdot \frac{s^l}{t^l}$$

But we know $f_0(t,s) = f_1(1/t, s/t)$ for all $t \neq 0$. Hence

$$\sum_{i=0}^m \sum_{j=0}^n a_{ij} t^i s^j = \sum_{k=0}^p \sum_{l=0}^r c_{kl} \cdot \frac{1}{t^k} \cdot \frac{s^l}{t^l}$$

We multiply across by t^{p+r} and see that

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}t^{i+p+r}s^j = \sum_{k=0}^{p}\sum_{l=0}^{r} c_{kl}s^l t^{p+r-k-l}$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}t^{i+p+r}s^j - \sum_{k=0}^{p}\sum_{l=0}^{r} c_{kl}s^l t^{p+r-k-l} = 0$$

Lemma 5.67 tells us this is the zero polynomial.

$$\begin{aligned}\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}t^{i+p+r}s^j &= a_{00}t^{p+r} + a_{01}t^{p+r}s + \cdots + a_{0n}t^{p+r}s^n \\ &+ a_{10}t^{p+r+1} + a_{11}t^{p+r+1}s + \cdots + a_{1n}t^{p+r+1}s^n \\ &+ a_{20}t^{p+r+2} + a_{21}t^{p+r+2}s + a_{22}t^{p+r+2}s^2 \cdots + a_{2n}t^{p+r+2}s^n \\ &\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ &+ a_{m0}t^{p+r+m} + a_{m1}t^{p+r+m}s + a_{m2}t^{p+r+2}s^2 \cdots + a_{mn}t^{p+r+m}s^n\end{aligned}$$

We see no two terms in this sum occur with the same degree of t and s.

$$\begin{aligned}\sum_{k=0}^{p}\sum_{l=0}^{r} c_{kl}s^l t^{p+r-k-l} &= c_{00}t^{p+r} + c_{01}st^{p+r-1} + \cdots + c_{0r}s^r t^p \\ &+ c_{10}t^{p+r-1} + c_{11}st^{p+r-2} + \cdots + c_{1r}s^r t^{p-1} \\ &+ c_{20}t^{p+r-2} + c_{21}st^{p+r-3} + c_{22}s^2t^{p+r-4} \cdots + c_{2r}s^r t^{p-2} \\ &\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ &+ c_{p0}t^r + c_{p1}st^{r-1} + c_{p2}s^2t^{r-2} + \cdots + c_{pr}s^r\end{aligned}$$

We see no two terms in this sum occur with the same degree of t and s. If $s^l t^{p+r-k-l} = s^{l'}t^{p+r-k'-l'}$ then $l = l'$ and hence $p+r-k-l = p+r-k'-l'$ gives $-k-l' = -k'-l'$ and hence $k = k'$.
We want to determine for which i, j, k and l is $t^{i+p+r}s^j = s^l t^{p+r-k-l}$. This means for which values is $i+p+r = p+r-k-l$ and $j = l$. Looking at $i+p+r = p+r-k-l$, we obtain $i = -k-l$ for i non negative. But k and l are also non negative. Hence $i = -k-l$ only when $i = k = l = 0$. Therefore $j = 0$. Therefore in the zero polynomial

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}t^{i+p+r}s^j - \sum_{k=0}^{p}\sum_{l=0}^{r} c_{kl}s^l t^{p+r-k-l}$$

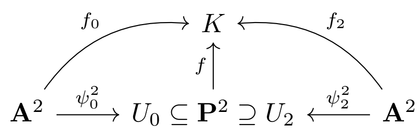
each $a_{ij} = 0$ for $i \neq 0$ or $j \neq 0$, each $c_{kl} = 0$ for $k \neq 0$ and $l \neq 0$. Therefore

$$f_0(t,s) = a_{00}$$

and

$$f_1(1/t, s/t) = c_{00}$$

are constant functions. But $a_{00} - c_{00} = 0$. Therefore $a_{00} = c_{00}$, hence $f_0(t, s) = f_1(1/t, s/t) = a_{00}$ are the same constant function.



Let $Q \in U_0 \cap U_2$. i.e $Q = [1 : t : s], s \neq 0$. Then $Q = \psi_0^2(t, s) = \psi_2^2(1/s, t/s)$. Hence $f(Q) = f(\psi_0^2(t, s)) = f_0(t, s) = a_{00}$. Also $f(Q) = f(\psi_1^2(1/s, t/s)) = f_2(1/s, t/s)$. Hence $f_0(t, s) = f_2(1/s, t/s)$ for all $s \neq 0$.

$f_0 \in K[y_1, y_2]$ hence we can express it as $f_0(y_1, y_2) = a_{00}$.
$f_2 \in K[y_0, y_1]$ hence we can express it as $f_2(y_0, y_1) = \sum_{k=0}^{w} \sum_{l=0}^{z} e_{kl} y_0^k y_1^l$ with $e_{kl} \in K$. Therefore

$$f_2(\frac{1}{s}, \frac{t}{s}) = \sum_{k=0}^{w} \sum_{l=0}^{z} e_{kl} \cdot \frac{1}{s^k} \cdot \frac{t^l}{s^l}$$

But we know $f_0(t, s) = f_2(1/s, t/s)$ for all $s \neq 0$. Hence

$$a_{00} = \sum_{k=0}^{w} \sum_{l=0}^{z} e_{kl} \cdot \frac{1}{s^k} \cdot \frac{t^l}{s^l}$$

We multiply across by s^{w+z} and see that

$$a_{00} s^{w+z} = \sum_{k=0}^{w} \sum_{l=0}^{z} e_{kl} s^{w+z-k-l} t^l$$

$$a_{00} s^{w+z} - \sum_{k=0}^{w} \sum_{l=0}^{z} e_{kl} s^{w+z-k-l} t^l = 0$$

Lemma 5.67 tells us this is the zero polynomial.

$$\begin{aligned}
\sum_{k=0}^{w} \sum_{l=0}^{z} e_{kl} s^{w+z-k-l} t^l &= e_{00} s^{w+z} + e_{01} s^{w+z-1} t + e_{02} s^{w+z-2} t^2 + \cdots + e_{0z} s^w t^z \\
&+ e_{10} s^{w+z-1} + e_{11} s^{w+z-2} t + e_{12} s^{w+z-3} t^2 + \cdots + e_{1z} s^{w-1} t^z \\
&+ e_{20} s^{w+z-2} + e_{21} s^{w+z-3} t + e_{22} s^{w+z-4} t^2 \cdots + e_{2z} s^{w-2} t^z \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
&+ e_{w0} s^z + e_{w1} s^{z-1} t + e_{w2} s^{z-2} t^2 + \cdots + e_{wz} t^z
\end{aligned}$$

We see no two terms in this sum occur with the same degree of t and s. If $s^{w+z-k-l}t^l = s^{w+z-k'-l'}t^{l'}$ then $l = l'$ and hence $w+z-k-l = w+z-k'-l'$ gives $-k-l' = -k'-l'$ and hence $k = k'$.
We want to determine for which k and l is $s^{w+z} = s^{w+z-k-l}t^l$. We can see that $l = 0$. So we check for which values is $w+z = w+z-k$. Looking at $w+z = w+z-k$, we obtain $0 = -k$, which give $k = 0$. Therefore in the zero polynomial

$$a_{00}s^{w+z} - \sum_{k=0}^{w}\sum_{l=0}^{z} e_{kl}s^{w+z-k-l}t^l$$

each $e_{kl} = 0$ for $k \neq 0$ or $l \neq 0$.

$$f_0(t,s) = a_{00}$$

and

$$f_2(1/s, t/s) = e_{00}$$

are constant functions. But $a_{00} - e_{00} = 0$. Therefore $a_{00} = e_{00}$. Hence $f_0(t,s) = f_2(1/s, t/s) = a_{00}$ are the same constant function.
If $P \in U_0$, then $P = \psi_0^2(t,s)$ for $(t,s) \in \mathbf{A}^2$. Hence $f(P) = f(\psi_0^2(t,s)) = f_0(t,s) = a_{00}$.
If $P \in U_1$, then $P = \psi_1^2(r,q)$ for $(r,q) \in \mathbf{A}^2$. Hence $f(P) = f(\psi_1^2(r,q)) = f_1(r,q) = a_{00}$.
If $P \in U_2$, then $P = \psi_2^2(e,g)$ for $(e,g) \in \mathbf{A}^2$. Hence $f(P) = f(\psi_2^2(e,g)) = f_2(e,g) = a_{00}$.

Hence f is constant on all of $\mathbf{P}^2$ as required.

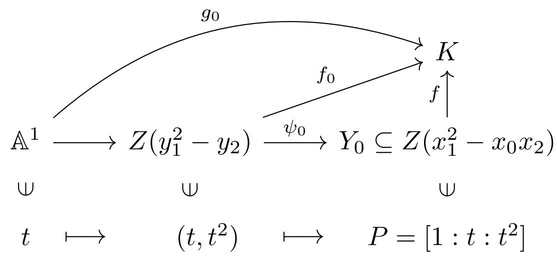
Example 5.69. Let $Y = Z(x_1^2 - x_0x_2) \subseteq \mathbf{P}^2$. We would like to show the only regular functions on $Z(x_1^2 - x_0x_2)$ are the constant functions.
Let $f \in \mathcal{O}(Z(x_1^2 - x_0x_2))$ be a global regular function.

$Y_0 = Z(x_1^2 - x_0x_2) \cap U_0 = \{P \in (x_1^2 - x_0x_2) : x_0 \neq 0\}$. The only point in $(x_1^2 - x_0x_2)$ but not in U_0 is $[0:0:1]$. Hence Y_0 is all of $(x_1^2 - x_0x_2)$ except $[0:0:1]$.

$Y_1 = Z(x_1^2 - x_0x_2) \cap U_1 = \{P \in (x_1^2 - x_0x_2) : x_1 \neq 0\}$. The only points in $(x_1^2 - x_0x_2)$ but not in U_1 are $[0:0:1]$ and $[1:0:0]$. Hence Y_1 is all of $(x_1^2 - x_0x_2)$ except $[0:0:1]$ and $[1:0:0]$.

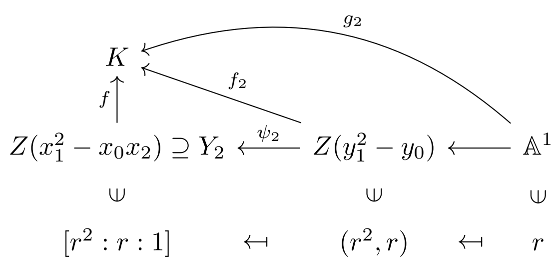
$Y_2 = Z(x_1^2 - x_0x_2) \cap U_2 = \{P \in (x_1^2 - x_0x_2) : x_2 \neq 0\}$. The only point in $(x_1^2 - x_0x_2)$ but not in U_2 is $[1:0:0]$. Hence Y_2 is all of $(x_1^2 - x_0x_2)$ except $[1:0:0]$.

Therefore Y_0 and Y_2 covers all of $Z(x_1^2 - x_0x_2)$ and we need only look at the maps in the following diagrams.



We see

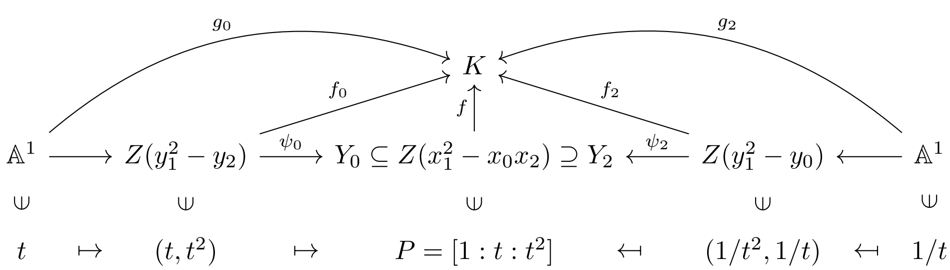
$$f_0 := \psi_0^*(f|_{Y_0})$$



We see

$$f_2 := (\psi_2)^*(f|_{Y_2})$$

Let $P = [1 : t : s] \in Y_0 \cap Y_2$ for $s \neq 0$ and $s = t^2$. Hence we have $P = [1 : t : t^2] = [1/t^2 : 1/t : 1]$ for $t \neq 0$. $f(P) = f_0(t, t^2) = f_2(1/t^2, 1/t)$. Our map now looks as follows:



Consider f_0 on the affine variety $Z(y_1^2 - y_2)$. Viewing f_0 as a polynomial, we have for the regular function $f_0 \in \mathcal{O}(Z(y_1^2 - y_2))$ a polynomial $\tilde{f_0} \in K[y_1, y_2]$ such that $\tilde{f_0}|_{Z(y_1^2 - y_2)} = f_0$. For all $(a_1, a_2) \in Z(y_1^2 - y_2), f_0(a_1, a_2) = \tilde{f_0}(a_1, a_2)$. In particular, for $a_1 = t$ and $a_2 = t^2$, $f_0(t, t^2) = \tilde{f_0}(t, t^2)$.

$A(Z(y_1^2-y_2)) = \frac{K[y_1,y_2]}{I(Z(y_1^2-y_2))}$. By Lemma 2.69, $\frac{K[y_1,y_2]}{(y_1^2-y_2)}$ is an integral domain, hence $(y_1^2-y_2)$ is a prime ideal generated by $y_1^2-y_2$.. Let $\mathfrak{a} = (y_1^2-y_2)$.

$$\begin{aligned} A(Z(y_1^2-y_2)) &= \frac{K[y_1,y_2]}{I(Z(y_1^2-y_2))} \\ &= \frac{K[y_1,y_2]}{I(Z(\mathfrak{a}))} \\ &= \frac{K[y_1,y_2]}{\sqrt{\mathfrak{a}}} \\ &= \frac{K[y_1,y_2]}{\mathfrak{a}} \\ &= \frac{K[y_1,y_2]}{(y_1^2-y_2)} \\ &= \frac{K[y_1,y_2]}{y_1^2-y_2} \end{aligned}$$

Put $g_0(t) = \tilde{f}_0(t,t^2)$. Then $g_0(t) \in K[t]$ is a polynomial in one variable.

Consider f_2 on the affine variety $Z(y_1^2-y_0)$. For $f_2 \in \mathcal{O}(Z(y_1^2-y_0))$ there is a polynomial $\tilde{f}_2 \in K[y_0,y_1]$ such that $\tilde{f}_2|_{Z(y_1^2-y_0)} = f_2$. For all $(a_1,a_2) \in Z(y_1^2-y_0), f_0(a_1,a_2) = \tilde{f}_0(a_1,a_2)$. In particular, for $a_1 = r^2$ and $a_2 = r$, $f_0(r^2,r) = \tilde{f}_0(r^2,r)$.
$A(Z(y_1^2-y_0)) = \frac{K[y_1,y_2]}{I(Z(y_1^2-y_0))}$. $\frac{K[y_1,y_2]}{(y_1^2-y_2)}$ is an integral domain, so $(y_1^2-y_0)$ is a prime ideal generated by $y_1^2-y_0$. Let $\mathfrak{b} = (y_1^2-y_0)$.

$$\begin{aligned} A(Z(y_1^2-y_0)) &= \frac{K[y_0,y_1]}{I(Z(y_1^2-y_0))} \\ &= \frac{K[y_0,y_1]}{I(Z(\mathfrak{b}))} \\ &= \frac{K[y_0,y_1]}{\sqrt{\mathfrak{b}}} \\ &= \frac{K[y_0,y_1]}{\mathfrak{b}} \\ &= \frac{K[y_0,y_1]}{(y_1^2-y_0)} \\ &= \frac{K[y_0,y_1]}{y_1^2-y_0} \end{aligned}$$

Put $\tilde{f}_2(r^2,r) = g_2(r)$. g_2 is a polynomial in one variable. For $P = [1:t:t^2]$, we now have

$$\begin{aligned} f(P) &= f[1:t:t^2] \\ &= f_0(t,t^2) \\ &= \tilde{f}_0(t,t^2) \\ &= g_0(t) \\ f(P) &= f[1/t^2 : t/t^2 : 1] \\ &= f_2(1/t^2, 1/t) \\ &= \tilde{f}_2(1/t^2, 1/t) \\ &= g_2(1/t) \end{aligned}$$

Hence $g_0(t) = g_2(1/t)$ for all $t \neq 0$. By Lemma 5.65 g_0 and g_2 are the same constant function. We call this constant a_0.

If $Q \in Y_0$, then $Q = \psi_0(t,t^2)$. Hence $f(Q) = f(\psi_0(t,t^2)) = f_0(t,t^2) = \tilde{f}_0(t,t^2) = g_0(t)$.

If $Q \in Y_2$, then $Q = \psi_2(r^2,r)$. Hence $f(Q) = f(\psi_2(r^2,r)) = f_2(r^2,r) = \tilde{f}_2(r^2,r) = g_2(r)$. But $g_0 = g_2 = a_0$. Hence $f = a_0$ Hence f is constant for all $P \in Z(x_1^2 - x_0x_2)$ as required.

We will show now that two affine varieties X and Y are isomorphic if and only if their coordinate rings are isomorphic as K-algebras.

Lemma 5.70. *Let X be any variety. $\mathcal{O}(X)$ is a k-algebra.*

Proof. $\mathcal{O}(X)$ is the ring of regular functions on X. It has point-wise addition and multiplication. By Lemma 5.4 if f, g is regular then $f+g$ and fg are regular. We define a multiplication of elements of $\mathcal{O}(X)$ by elements of K as follows:

$$K \times \mathcal{O}(X) \to \mathcal{O}(X)$$

$$(k, f) \mapsto kf$$

where $(kf)x = kf(x)$ for all $x \in X$. We must check that kf is regular. Let X be a quasi-affine variety. We are given that f is regular on X. This means for each $P \in X$ there is an open neighbourhood $U \subseteq X$ with $P \in U$,$f = \frac{g}{h}$ on U for $g, h \in K[y_1, \ldots, y_n]$ and $h \neq 0$ on U. We can take the same U and guess $kf = \frac{kg}{h}$. $kg \in K[y_1, \ldots, y_n]$ and $h \neq 0$ on U. Therefore kf is regular on X.
Let X be a quasi-projective variety. We are given that f is regular on X. This means for each $Q \in X$ there is an open neighbourhood $V \subseteq X$ with $Q \in V$,$f = \frac{g}{h}$ on V for $g, h \in K[x_0, \ldots, x_n]$ homogeneous of same degree and $h \neq 0$ on V. We can take the same V and guess $kf = \frac{kg}{h}$. $kg \in K[x_0, \ldots, x_n]$, kg, h homogeneous of same degree and $h \neq 0$ on V. Therefore kf is regular on X.
Let $1 \in K$ be the unit element and $f \in \mathcal{O}(X)$.Then $(1f)x = 1f(x) = f(x)$. Let

$k_1, k_2 \in K$ and $f, g \in \mathcal{O}(X)$.

$$\begin{aligned}(k_1(f+g))(x) &= k_1((f+g)x) \\ &= k_1(f(x)+g(x)) \\ &= k_1(f(x)) + k_1(g(x)) \\ &= (k_1 f)(x) + (k_1 g)(x) \\ &= (k_1 f + k_1 g)(x)\end{aligned}$$

We see $(k_1(f+g))(x) = (k_1 f + k_1 g)(x)$, hence $k_1(f+g) = k_1 f + k_1 g$.

$$\begin{aligned}((k_1+k_2)f)(x) &= (k_1+k_2)f(x) \\ &= k_1 f(x) + k_2 f(x) \\ &= (k_1 f)(x) + (k_2 f)(x) \\ &= (k_1 f + k_2 f)(x)\end{aligned}$$

We see $((k_1+k_2)f)(x) = (k_1 f + k_2 f)(x)$, hence $(k_1+k_2)f = k_1 f + k_2 f$

$$\begin{aligned}(k_1(k_2 f))(x) &= k_1((k_2 f)(x)) \\ &= k_1(k_2 f(x)) \\ &= (k_1 k_2) f(x)) \\ &= ((k_1 k_2) f)(x)\end{aligned}$$

We see $(k_1(k_2 f))(x) = ((k_1 k_2) f)(x)$, hence $k_1(k_2 f) = (k_1 k_2) f$ Therefore $\mathcal{O}(X)$ is a K vector space.
Let $k \in K$ and $f, g \in \mathcal{O}(X)$. Want to show

$$k(fg) = (kf)g = f(kg)$$

$$\begin{aligned}(k(fg))(x) &= k((fg)(x)) \\ &= k(f(x)g(x)) \\ &= (kf(x))g(x) \\ &= ((kf)(x))g(x) \\ &= ((kf)g)(x)\end{aligned}$$

We see $(k(fg))(x) = ((kf)g)(x)$. Hence $k(fg) = (kf)g$. Continuing on we obtain,

$$\begin{aligned}((kf)g)(x) &= (kf)(x)g(x) \\ &= (kf(x))g(x) \\ &= (f(x)k)g(x) \\ &= f(x)(kg(x) \\ &= f(x)((kg)(x)) \\ &= (f(kg))(x)\end{aligned}$$

We see $((kf)g)(x) = (f(kg))(x)$.Hence $(kf)g = f(kg)$. Therefore $k(fg) = (kf)g = f(kg)$. Hence $\mathcal{O}(X)$ is a K-algebra. $\square$

By Lemma 2.66 for an affine variety Y, $A(Y)$ is a K-algebra.

Lemma 5.71. *Let X be any variety, and let $Y \subseteq \mathbf{A}^n$ be an affine variety. A map of sets $\gamma : X \to Y$ is a morphism if and only if $x_i \circ \gamma$ is a regular function on X for each i, where $x_1, ..., x_n$ are the coordinate functions on $\mathbf{A}^n$.*

Proof. Let γ be a morphism. Then for each open $V \subset Y$ and regular function $f : V \to K$, $f \circ \gamma$ is regular. In particular for $f = x_i$, $\quad x_i \circ \gamma$ is regular.
Assume the $x_i \circ \gamma$ are regular. i.e $x_1 \circ \gamma, x_2 \circ \gamma \ldots, x_n \circ \gamma$ are all regular on X. Any multiplication, addition and scalar multiplication of regular functions on X is again regular on X. Hence for any $f \in K[x_1, \ldots, x_n], f \circ \gamma$ is regular on X. Let C be a closed subset of Y. Then $C = Z(T)$ where T is a set of polynomials. i.e $C = Z(f_1, f_2, \ldots, f_r, \ldots)$ for $f_1, f_2, \ldots, f_r, \cdots \in K[x_1, x_2 \ldots, x_n]$. $Z(f_1, f_2, \ldots, f_r, \ldots) = \{P \in \mathbf{A}^n : f_1(P) = f_2(P) = \cdots = f_r(P) = 0 = \ldots\}$. Let $Q \in \gamma^{-1}(C)$. Then $\gamma(Q) \in C$. So we have

$$\begin{aligned}f_1(\gamma(Q)) = f_2(\gamma(Q)) = \cdots = f_r(\gamma(Q)) = \cdots = 0 \\ (f_1 \circ \gamma)(Q) = (f_2 \circ \gamma)(Q) = \cdots = (f_r \circ \gamma)(Q) = \cdots = 0\end{aligned}$$

Hence $\gamma^{-1}(C) \subseteq Z(f_1 \circ \gamma, f_2 \circ \gamma, \ldots, f_r \circ \gamma, \ldots)$.
Let $R \in Z(f_1 \circ \gamma, f_2 \circ \gamma, \ldots, f_r \circ \gamma \ldots)$. Then,

$$\begin{aligned}(f_1 \circ \gamma)(R) = (f_2 \circ \gamma)(R) = \cdots = (f_r \circ \gamma)(R) = \cdots = 0 \\ f_1(\gamma(R)) = f_2(\gamma(R)) = \cdots = f_r(\gamma(R)) = \cdots = 0\end{aligned}$$

Therefore $\gamma(R) \in C$. Hence $R \in \gamma^{-1}(C)$. Therefore $Z(f_1 \circ \gamma, f_2 \circ \gamma, \ldots, f_r \circ \gamma \ldots) \subseteq \gamma^{-1}(C)$. Hence $\gamma^{-1}(C) = Z(f_1 \circ \gamma, f_2 \circ \gamma, \ldots, f_r \circ \gamma, \ldots)$. By Lemma 5.9 $Z(f_1 \circ \gamma, f_2 \circ \gamma, \ldots, f_r \circ \gamma \ldots)$ is closed. Hence $\gamma^{-1}(C)$ is closed in X. This shows γ is a continuous map. Let $V \subset Y$ be open and $g : V \to K$ be a regular function. Let $P \in \gamma^{-1}(V)$ then $\gamma(P) \in V$. Since g is regular, there is an open neighbourhood $W \subseteq V$ with $\gamma(P) \in W$ and $h, j \in K[x_1, \ldots x_n]$ such that

$g = h/j$ with $j \neq 0$ on W. This gives us that $g(\gamma(P)) = h(\gamma(P))/j(\gamma(P))$. Let $P' \in \gamma^{-1}(W)$. $\gamma^{-1}(W)$ is open in $\gamma^{-1}(V)$. $(g \circ \gamma)(P') = g(\gamma(P'))$. Since g is regular $g(\gamma(P')) = h(\gamma(P')/j(\gamma(P') = (h \circ \gamma)(P')/(j \circ \gamma)(P')$. Therefore $(g \circ \gamma)(P') = (h \circ \gamma)(P')/(j \circ \gamma)(P')$. By Lemma 5.4 $(g \circ \gamma)$ is regular. Hence γ is a morphism. □

Theorem 5.72. *Let X be any variety and Y an affine variety. There is a natural bijective mapping of sets*

$$\alpha : \ Hom\ (X, Y) \hookrightarrow \ Hom\ (A(Y), \mathcal{O}(X))$$

where $Hom\ (X, Y)$ is the set of morphisms of varieties and $Hom\ (A(Y), \mathcal{O}(X))$ is the set of homomorphisms of K-algebras.

Proof. Let $\phi : X \to Y$ be a morphism. Then ϕ induces a homomorphism of K-algebras,

$$\phi^{\mathcal{O}} : \mathcal{O}(Y) \to \mathcal{O}(X)$$

$$f \mapsto f \circ \phi$$

where $f \circ \phi$ is regular on $\phi^{-1}(Y) = X$. Clearly $\phi^{-1}(Y) \subseteq X$. Let $x \in X$. Then $\phi(x) = y$ for some $y \in Y$. Therefore $\phi^{-1}(Y) \supseteq X$. Hence $\phi^{-1}(Y) = X$. $\phi^{\mathcal{O}}$ is actually a homomorphism of K-algebras. Let $k \in K$, $f \in \mathcal{O}(Y)$ and $x \in X$. $\phi^{\mathcal{O}}(kf) = kf \circ \phi$. Hence

$$\begin{aligned}((kf) \circ \phi)(x) &= (kf)(\phi(x)) \\ &= k(f(\phi(x))) \\ &= k((f \circ \phi)(x)) \\ &= k((\phi^{\mathcal{O}}(f))(x)) \\ &= (k\phi^{\mathcal{O}}(f))(x)\end{aligned}$$

We see $(\phi^{\mathcal{O}}(kf))(x) = (k\phi^{\mathcal{O}}(f))(x)$. Hence $\phi^{\mathcal{O}}(kf) = k(\phi^{\mathcal{O}}(f))$.
By Theorem 5.43 part (a) $\tau_Y : A(Y) \cong \mathcal{O}(Y)$. Hence we have a homomorphism $h : A(Y) \to \mathcal{O}(Y)$. Let $k \in K$, and $f + I(Y) \in A(Y)$.

$$\begin{aligned}h(k(f + I(Y))) &= h(kf + I(Y)) \\ &= (kf)_{|Y} \\ &= k(f_{|Y}) \\ &= kh(f + I(Y))\end{aligned}$$

Hence τ_Y is actually a homomorphism of K-algebras. By Lemma 2.64
$\phi^{\mathcal{O}} \circ \tau_Y : A(Y) \to \mathcal{O}(X)$ is a homomorphism of K-algebras.
Hence the map α is defined as $\alpha(\phi) = \phi^{\mathcal{O}} \circ \tau_Y$

Let g be a homomorphism of K-algebras $A(Y) \to \mathcal{O}(X)$. We take Y to be a closed subset of $\mathbf{A}^n$. Recall $A(Y) = K[x_1, \ldots, x_n]/I(Y)$.. For each $x_i \in A = K[x_1, \ldots, x_n]$ we let $\overline{x}_i \in A(Y)$ be the image of x_i. Then each $\xi_i = g(\overline{x}_i) \in \mathcal{O}(X)$ is a global regular function on X. We define a map $\gamma : X \to \mathbf{A}^n$ as $\gamma(P) = (\xi_1(P), \xi_2(P), \ldots, \xi_n(P))$. Y is a closed subset of $\mathbf{A}^n$ hence $Y = Z(I(Y))$. Let $f \in K[x_1, \ldots x_n]$. Then $f(\gamma(P)) = f(\xi_1(P), \xi_2(P), \ldots, \xi_n(P))$. But $\xi_i = g(\overline{x}_i)$ hence

$$f(\gamma(P)) = f(g(\overline{x}_1)(P), g(\overline{x}_2)(P), \ldots, g(\overline{x}_n)(P))$$

We show $f(g(\overline{x}_1)(P), g(\overline{x}_2)(P), \ldots, g(\overline{x}_n)(P)) = g(f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n))(P)$.
Let f be a constant function c. Then

$$\begin{aligned} g(f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n))(P) &= g(c \cdot 1_{A(Y)})(P) \\ &= (cg(1_{A(Y)}))(P) \\ &= (c(1_{\mathcal{O}(X)}))(P) \\ &= c \cdot 1_K \end{aligned}$$

$$\begin{aligned} f(g(\overline{x}_1)(P), g(\overline{x}_2)(P), \ldots, g(\overline{x}_n)(P)) &= f(k_1, \ldots, k_n) \text{ for } k_1, \ldots k_n \in K \\ &= c \cdot 1_K \end{aligned}$$

Let $f(x_1, \ldots x_n) = x_i$.

$$\begin{aligned} g(f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n))(P) &= g(f + I(Y))(P) \\ &= g(x_i + I(Y))(P) \\ &= g(\overline{x}_i)(P) \end{aligned}$$

$$f(g(\overline{x}_1)(P), g(\overline{x}_2)(P), \ldots, g(\overline{x}_n)(P)) = g(\overline{x}_i)(P)$$

Let $p, q \in K[x_1, \ldots, x_n]$ hold. Put $f = p + q$. Then

$$\begin{aligned} g(f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n))(P) &= g(f + I(Y))(P) \\ &= g(p + q + I(Y))(P) \\ &= g(\overline{p} + \overline{q})(P) \end{aligned}$$

$$\begin{aligned} f(g(\overline{x}_1)(P), g(\overline{x}_2)(P), \ldots, g(\overline{x}_n)(P)) &= g(\overline{p})(P) + g(\overline{q})(P) \\ &= g(\overline{p} + \overline{q})(P) \end{aligned}$$

Put $f = pq$. Then

$$\begin{aligned} g(f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n))(P) &= g(f + I(Y))(P) \\ &= g(pq + I(Y))(P) \\ &= g(\overline{pq})(P) \end{aligned}$$

$$\begin{aligned} f(g(\overline{x}_1)(P), g(\overline{x}_2)(P), \ldots, g(\overline{x}_n)(P)) &= g(\overline{p})(P) g(\overline{q})(P) \\ &= g(\overline{pq})(P) \end{aligned}$$

So we see it holds for $f = c, f = x_i, f = p + q$ and $f = pq$. Hence the equation will hold for all $f \in A$. Therefore $f(g(\overline{x}_1)(P), g(\overline{x}_2)(P), \ldots, g(\overline{x}_n)(P)) = g(f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n))(P)$. In particular if $f \in I(Y)$ then

$$\begin{aligned} g(f + I(Y))(P) &= g(0 + I(Y))(P) \\ &= (0_{\mathcal{O}(X)})(P) \\ &= 0_K \end{aligned}$$

Hence $f(g(\overline{x}_1)(P), g(\overline{x}_2)(P), \ldots, g(\overline{x}_n)(P)) = g(f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n))(P) = 0$. Therefore $f(g(\overline{x}_1)(P), g(\overline{x}_2)(P), \ldots, g(\overline{x}_n)(P)) = 0$. Hence $f(\gamma(P)) = 0$ for all $P \in X$. Therefore $Im(\gamma) \subseteq Y$. Hence γ is a map from X to Y. γ is a morphism by Lemma 5.71. We will now show $\alpha(\gamma)) = g$.
By definition $\alpha(\gamma) = \gamma^{\mathcal{O}} \circ \tau_Y$. Let $\overline{f} \in A(Y)$ and $P \in X$. Then

$$\begin{aligned} (\gamma^{\mathcal{O}} \circ \tau_Y)(\overline{f}) &= \gamma^{\mathcal{O}}(\tau_Y(\overline{f})) \\ &= \gamma^{\mathcal{O}}(\overline{f}|_Y) \\ &= \overline{f}|_Y \circ \gamma \end{aligned}$$

Taking the point $P \in X$ we have

$$\begin{aligned} (\overline{f}|_Y \circ \gamma)(P) &= \overline{f}|_Y(\gamma(P)) \\ &= (\overline{f}|_Y))(g(\overline{x}_1)(P), g(\overline{x}_2)(P), \ldots, g(\overline{x}_n)(P)) \\ &= (\overline{f}))(g(\overline{x}_1)(P), g(\overline{x}_2)(P), \ldots, g(\overline{x}_n)(P)) \\ &= g(\overline{f}(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n))(P) \\ &= (g(\overline{f}))(P) \end{aligned}$$

Hence γ induces the homomorphism g. Therefore α is surjective.

Let $\phi, \varphi \in Hom(X,Y)$ such that $\alpha(\phi) = \alpha(\varphi)$. Then $\phi^{\mathcal{O}} \circ \tau_Y = \varphi^{\mathcal{O}} \circ \tau_Y$. Hence $\phi^{\mathcal{O}} = \varphi^{\mathcal{O}}$. Let $f \in \mathcal{O}(Y)$. Then $\phi^{\mathcal{O}}(f) = \varphi^{\mathcal{O}}(f)$. Therefore $f \circ \phi = f \circ \varphi$. Hence $\phi = \varphi$. Therefore α is injective.

α is a bijective map. □

Lemma 5.73. *Let X, Y be affine varieties. Then*

$$\alpha' : \; Hom\ (X,Y) \hookrightarrow \; Hom\ (A(Y), A(X))$$

where Hom (X,Y) is the set of morphisms of varieties and Hom $(A(Y), A(X))$ is the set of homomorphisms of K-algebras is bijective.

Proof. We have $\alpha : Hom(X,Y) \to Hom(A(Y), \mathcal{O}(X))$ is bijective. We will show the map

$$\begin{aligned} \beta : Hom(A(Y), \mathcal{O}(X)) &\to Hom(A(Y), A(X) \\ g &\mapsto \tau_X^{-1} \circ g \end{aligned}$$

is bijective. The map $\beta' : Hom(A(Y), A(X)) \to Hom(A(Y), \mathcal{O}(X))$ is defined by $h \mapsto \tau_X \circ h$. We will show β' is the inverse map of β.

Let $g \in Hom(A(Y), \mathcal{O}(X))$.

$$\begin{aligned} (\beta' \circ \beta)(g) &= \beta'(\beta(g)) \\ &= \beta'(\tau_X^{-1} \circ g) \\ &= \tau_X \circ \tau_X^{-1} \circ g \\ &= g \end{aligned}$$

Let $h \in Hom(A(Y), A(X))$.

$$\begin{aligned} (\beta \circ \beta')(h) &= \beta(\beta'(h)) \\ &= \beta(\tau_X \circ h) \\ &= \tau_X^{-1} \circ \tau_X \circ h \\ &= h \end{aligned}$$

Hence β is bijective. The composition of two bijective maps is bijective. Hence we have $\beta \circ \alpha$ is bijective. $\beta \circ \alpha$ is the map $Hom(X,Y) \to Hom(A(Y), A(X))$ defined by $\phi \mapsto \phi^A$. □

Lemma 5.74. *Let $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ be affine varieties with $Y' \subseteq Y$ either an open subset or irreducible closed subset. Let $\phi : X \to Y'$ be a map. ϕ is a morphism if and only if $i \circ \phi : X \to Y$ is a morphism where $i : Y' \to Y$ is the inclusion map.*

Proof. Let ϕ be a morphism. By Lemma 5.21 i is a morphism. By Lemma 5.19, $i \circ \phi$ is a morphism.

Conversely, let $i \circ \phi$ be a morphism.
Case 1: When Y' is an open subset of Y. Let $O \subset Y'$ be an open subset. By the subspace topology on Y', $O = V \cap Y'$ for some open $V \subseteq Y$. Y' and V are open in Y hence $V \cap Y'$ is open in Y. Hence O is open in Y. Therefore $(i \circ \phi)^{-1}(O)$ is open in X. $(i \circ \phi)^{-1}(O) = \phi^{-1}(i^{-1}(O)) = \phi^{-1}(O)$. Hence $\phi^{-1}(O)$ is open in X showing ϕ is a continuous map.
Let $O \subseteq Y'$ be an open subset and $f : O \to K$ regular. By the subspace topology on Y', $O = Y' \cap V$ for some open $V \subseteq Y$. Therefore $Y' \cap V$ is open in Y. Therefore O is open in Y. Hence we have that $f \circ i \circ \phi : (i \circ \phi)^{-1}(O) \to K$ is regular. Let $P \in (i \circ \phi)^{-1}(O)$. $(i \circ \phi)^{-1}(O) = \phi^{-1}(i^{-1}(O)) = \phi^{-1}(O)$. Hence $P \in \phi^{-1}(O)$.

$$\begin{aligned}(f \circ i \circ \phi)(P) &= (f \circ i)(\phi(P)) \\ &= f(i(\phi(P)) \\ &= f(\phi(P)) \\ &= (f \circ \phi)(P)\end{aligned}$$

Therefore $f \circ i \circ \phi = f \circ \phi$. Hence we see $f \circ \phi : \phi^{-1}(O) \to K$ is regular.

Case 2: Let Y' be an irreducible closed subset. This means Y' is an affine variety. We have assumed $i \circ \phi$ is a morphism. Hence by Lemma 5.71, $x_1 \circ i \circ \phi, x_2 \circ i \circ \phi, \dots x_m \circ i \circ \phi$ are regular on X. Let $P \in X$. For each $j \in 1, \dots, m$ $(x_j \circ i \circ \phi)(P) = (x_j \circ i)(\phi(P)) = (x_j \circ \phi)(P)$. Hence $x_j \circ i \circ \phi$ and $x_j \circ \phi$ are the same map. Hence $x_j \circ \phi$ is regular on X.

By Lemma 5.71, ϕ is a morphism.

□

Example 5.75. Let X and Y be affine varieties and $\phi : X \to Y$ a morphism. By Theorem 5.72 there is a natural bijection between $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(A(Y), \mathcal{O}(X))$ defined by $\phi \mapsto \phi^{\mathcal{O}} \circ \tau_Y$. By Theorem 5.43 $\tau_X : A(X) \cong \mathcal{O}(X)$. Hence we have a K-algebra homorphism $\tau_X^{-1} \circ \phi^{\mathcal{O}} \circ \tau_Y : A(Y) \to A(X)$. We can see this more clearly in the following diagram.

$$\begin{array}{ccc} A(Y) & \xrightarrow{\phi^A} & A(X) \\ \big\downarrow{\scriptstyle \tau_Y} & & \big\downarrow{\scriptstyle \tau_X} \\ \mathcal{O}(Y) & \xrightarrow{\phi^{\mathcal{O}}} & \mathcal{O}(X) \end{array}$$

From the diagram, by definition, $\phi^A = \tau_X^{-1} \circ \phi^{\mathcal{O}} \circ \tau_Y$.

Lemma 5.76. *Let X, Y, Z be varieties. Let $\phi : X \to Y, \psi : Y \to Z$ be mor-*

phisms. ϕ induces

$$\begin{aligned}\phi^{\mathcal{O}} : \mathcal{O}(Y) &\to \mathcal{O}(X)\\ f &\mapsto f \circ \phi\end{aligned}$$

ψ induces

$$\begin{aligned}\psi^{\mathcal{O}} : \mathcal{O}(Z) &\to \mathcal{O}(Y).\\ g &\mapsto g \circ \psi\end{aligned}$$

Then $(\psi \circ \phi)^{\mathcal{O}} = \phi^{\mathcal{O}} \circ \psi^{\mathcal{O}}$.

Proof. Since ϕ and ψ are morphisms the composition $\psi \circ \phi$ is a morphism. $\psi \circ \phi$ induces

$$\begin{aligned}(\psi \circ \phi)^{\mathcal{O}} : \mathcal{O}(Z) &\to \mathcal{O}(X)\\ h &\mapsto h \circ (\psi \circ \phi)\end{aligned}$$

Let $g \in \mathcal{O}(Z)$. Then

$$\begin{aligned}(\phi^{\mathcal{O}} \circ \psi^{\mathcal{O}})(g) &= \phi^{\mathcal{O}}(\psi^{\mathcal{O}}(g))\\ &= \phi^{\mathcal{O}}(g \circ \psi)\\ &= g \circ \psi \circ \phi\\ &= g \circ (\psi \circ \phi)\\ &= (\psi \circ \phi)^{\mathcal{O}}(g)\end{aligned}$$

Hence $(\psi \circ \phi)^{\mathcal{O}} = \phi^{\mathcal{O}} \circ \psi^{\mathcal{O}}$. ☐

Lemma 5.77. *Let X, Y, Z be affine varieties. Let $\phi : X \to Y, \psi : Y \to Z$ be morphisms. ϕ induces*

$$\begin{aligned}\phi^{A} : A(Y) &\to A(X)\\ \overline{f} &\mapsto \overline{f} \circ \phi\end{aligned}$$

ψ induces

$$\begin{aligned}\psi^{A} : A(Z) &\to A(Y).\\ \overline{g} &\mapsto \overline{g} \circ \psi\end{aligned}$$

Then $(\psi \circ \phi)^{A} = \phi^{A} \circ \psi^{A}$.

Proof. Since ϕ and ψ are morphisms the composition $\psi \circ \phi$ is a morphism. $\psi \circ \phi$ induces

$$\begin{aligned}(\psi \circ \phi)^A : A(Z) &\to A(X)\\ \overline{h} &\mapsto \overline{h} \circ (\psi \circ \phi)\end{aligned}$$

Let $\overline{g} \in A(Z)$. Then

$$\begin{aligned}(\phi^A \circ \psi^A)(\overline{g}) &= \phi^A(\psi^A(\overline{g}))\\ &= \phi^A(\overline{g} \circ \psi)\\ &= \overline{g} \circ \psi \circ \phi\\ &= \overline{g} \circ (\psi \circ \phi)\\ &= (\psi \circ \phi)^A(\overline{g})\end{aligned}$$

Hence $(\psi \circ \phi)^A = \phi^A \circ \psi^A$. □

Theorem 5.78. *Let X and Y be affine varieties. X and Y are isomorphic if and only if $A(X)$ and $A(Y)$ are isomorphic as K-algebras.*

Proof. Let $\phi : X \mapsto Y$ be an isomorphism of affine varieties. This means ϕ is a morphism which admits an inverse morphism $\psi : Y \to X$ such that $\phi \circ \psi = \text{Id}_Y$ and $\psi \circ \phi = \text{Id}_X$. From Example 5.75 we have a K-algebra homomorphism $\phi^A = \tau_X^{-1} \circ \phi^{\mathcal{O}} \circ \tau_Y : A(Y) \to A(X)$.
Similarly since ψ is a morphism ψ induces a K-algebra homomorphism

$$\psi^{\mathcal{O}} : \mathcal{O}(X) \to \mathcal{O}(Y)$$

$$\begin{array}{ccc} A(X) & \xrightarrow{\psi^A} & A(Y) \\ \big\downarrow{\scriptstyle \tau_X} & & \big\downarrow{\scriptstyle \tau_Y} \\ \mathcal{O}(X) & \xrightarrow{\psi^{\mathcal{O}}} & \mathcal{O}(Y) \end{array}$$

From the diagram, by definition, $\psi^A = \tau_Y^{-1} \circ \psi^{\mathcal{O}} \circ \tau_X : A(X) \to A(Y)$.

$$
\begin{aligned}
\phi^A \circ \psi^A &= \tau_X^{-1} \circ \phi^{\mathcal{O}} \circ \tau_Y \circ \tau_Y^{-1} \circ \psi^{\mathcal{O}} \circ \tau_X \\
&= \tau_X^{-1} \circ \phi^{\mathcal{O}} \circ \psi^{\mathcal{O}} \circ \tau_X \\
&= \tau_X^{-1} \circ (\psi \circ \phi)^{\mathcal{O}} \circ \tau_X \\
&= \tau_X^{-1} \circ (id_X)^{\mathcal{O}} \circ \tau_X \\
&= \tau_X^{-1} \circ ((id_X)^{\mathcal{O}} \circ \tau_X) \\
&= \tau_X^{-1} \circ \tau_X \\
&= id_{A(X)}
\end{aligned}
$$

$$
\begin{aligned}
\psi^A \circ \phi^A &= \tau_Y^{-1} \circ \psi^{\mathcal{O}} \circ \tau_X \circ \tau_X^{-1} \circ \phi^{\mathcal{O}} \circ \tau_Y \\
&= \tau_Y^{-1} \circ \psi^{\mathcal{O}} \circ \phi^{\mathcal{O}} \circ \tau_Y \\
&= \tau_Y^{-1} \circ (\phi \circ \psi)^{\mathcal{O}} \circ \tau_Y \\
&= \tau_Y^{-1} \circ (id_Y)^{\mathcal{O}} \circ \tau_Y \\
&= \tau_Y^{-1} \circ ((id_Y)^{\mathcal{O}} \circ \tau_Y) \\
&= \tau_Y^{-1} \circ \tau_Y \\
&= id_{A(Y)}
\end{aligned}
$$

Hence $A(Y) \cong A(X)$.

Let $A(Y) \cong A(X)$. By definition of K-algebra isomorphism, $A(Y) \to A(X)$ is a bijective map with a homomorphism $g : A(Y) \to A(X)$ and an inverse homomorphism $g' : A(X) \to A(Y)$. Taking g first we recall that $\tau_X : A(X) \cong \mathcal{O}(X)$. Therefore we have a homomorphism $\tau_X \circ g : A(Y) \to \mathcal{O}(X)$. Hence by Theorem 5.72 there exists $\phi \in Hom(X, Y)$ such that $\alpha(\phi) = \tau_X \circ g$. But by definition $\alpha(\phi) = \phi^{\mathcal{O}} \circ \tau_Y$. Hence $\tau_X \circ g = \phi^{\mathcal{O}} \circ \tau_Y$. From this we obtain $g = \tau_X^{-1} \circ \phi^{\mathcal{O}} \circ \tau_Y$. Hence $g = \phi^A$. Given that $g' : A(X) \to A(Y)$ is also a homomorphism we obtain a homomorphism of K-algebras $\tau_Y \circ g' : A(X) \to \mathcal{O}(Y)$. Hence by Theorem 5.72 there exists $\psi \in Hom(Y, X)$ such that $\alpha(\psi) = \tau_Y \circ g'$. But by definition $\alpha(\psi) = \psi^{\mathcal{O}} \circ \tau_X$.Hence $\tau_Y \circ g' = \psi^{\mathcal{O}} \circ \tau_X$. From this we obtain $g' = \tau_Y^{-1} \circ \psi^{\mathcal{O}} \circ \tau_X$. Hence $g' = \psi^A$.

$$
\begin{aligned}
(\phi \circ \varphi)^A &= \varphi^A \circ \phi^A \\
&= g' \circ g \\
&= Id_{A(Y)}
\end{aligned}
$$

By Lemma 5.73 $\phi \circ \varphi = Id_Y$.

$$\begin{aligned}(\varphi \circ \phi)^A &= \phi^A \circ \varphi^A \\ &= g \circ g' \\ &= Id_{A(X)}\end{aligned}$$

By Lemma 5.73 $\varphi \circ \phi = Id_X$.

Hence $X \cong Y$. □

Example 5.79. Recall in Lemma 2.69 we proved $A(Y) \cong K[t]$ where $Y = Z(y - x^2)$. Theorem 5.78 tells us $Z(y - x^2) \cong \mathbf{A}^1$ since $K[t]$ is the coordinate ring of $\mathbf{A}^1$.

5.4.2 Examples of morphisms

Example 5.80. Let $\tau : \mathbf{A}^1 \to \mathbf{A}^2$ be the map defined by $t \mapsto (t, t^2)$. τ is a morphism.

Proof. Let $t \in \mathbf{A}^1$ and x the coordinate function on $\mathbf{A}^2$. $(x \circ \tau)(t) = t$. We have a map

$$\begin{aligned}x \circ \tau :&\mathbf{A}^1 \to K \\ &t \mapsto t\end{aligned}$$

We need for each $t \in \mathbf{A}^1$ an open $U \subseteq \mathbf{A}^1$ with $t \in U$ and $g, h \in K[r]$ with $h(r) \neq 0$ on U and $(x \circ \tau) = g(r)/h(r)$. We choose $U = \mathbf{A}^1$ and $(x \circ \tau) = g(r)/h(r)$ where $g(r) = t$ and $h(r) = 1$. Hence we see $x \circ \tau$ is regular.

Let $t \in \mathbf{A}^1$ and y the coordinate function on $\mathbf{A}^2$. $(y \circ \tau)(t) = t^2$. We have a map

$$\begin{aligned}y \circ \tau :&\mathbf{A}^1 \to K \\ &t \mapsto t^2\end{aligned}$$

We choose our open set to be $\mathbf{A}^1$. $(y \circ \tau) = g(r)/h(r)$ where $g(r) = t^2$ and $h(r) = 1$. Hence we see $y \circ \tau$ is regular. By Lemma 5.71 τ is a morphism. □

Example 5.81. Let $\gamma : \mathbf{A}^1 \to \mathbf{A}^2$ be the map defined by $s \mapsto (s^2, s)$. γ is a morphism.

Proof. Let $s \in \mathbf{A}^1$ and x the coordinate function on $\mathbf{A}^2$. $(x \circ \gamma)(s) = s^2$. We have a map

$$\begin{aligned}x \circ \gamma :&\mathbf{A}^1 \to K \\ &s \mapsto s^2\end{aligned}$$

We need for each $s \in \mathbf{A}^1$ an open $U \subseteq \mathbf{A}^1$ with $s \in U$ and $g, h \in K[r]$ with $h(r) \neq 0$ on U and $(x \circ \gamma) = g(r)/h(r)$. We choose $U = \mathbf{A}^1$ and $(x \circ \gamma) = g(r)/h(r)$ where $g(r) = s^2$ and $h(r) = 1$. Hence we see $x \circ \gamma$ is regular.

Let $s \in \mathbf{A}^1$ and y the coordinate function on $\mathbf{A}^2$. $(y \circ \gamma)(s) = s$. We have a map

$$\begin{aligned} y \circ \gamma : &\mathbf{A}^1 \to K \\ &s \mapsto s \end{aligned}$$

Again we choose our open set to be $\mathbf{A}^1$. $y \circ \gamma = g(r)/h(r)$ where $g(r) = s$ and $h(r) = 1$. Hence we see $y \circ \gamma$ is regular. By Lemma 5.71 γ is a morphism. ☐

Example 5.82. Let $\omega : \mathbf{A}^2 \to \mathbf{A}^1$ be the map defined by $(s, t) \mapsto s$. ω is a morphism.

Proof. Let $(s, t) \in \mathbf{A}^2$ and x the coordinate function on $\mathbf{A}^1$. $(x \circ \omega)(s, t) = s$. We have a map

$$\begin{aligned} x \circ \omega : &\mathbf{A}^2 \to K \\ &(s, t) \mapsto s \end{aligned}$$

We need for each $(s, t) \in \mathbf{A}^1$ an open $U \subseteq \mathbf{A}^2$ with $(s, t) \in U$ and $g, h \in K[r_1, r_2]$ with $h(r_1, r_2) \neq 0$ on U and $(x \circ \omega) = g(r_1, r_2)/h(r_1, r_2)$. We choose $U = \mathbf{A}^2$ and $x \circ \omega = g(r_1, r_2)/h(r_1, r_2)$ where $g(r_1, r_2) = s$ and $h(r_1, r_2) = 1$. Hence we see $x \circ \omega$ is regular on $\mathbf{A}^2$.

By Lemma 5.71 ω is a morphism. ☐

Example 5.83. Let $\eta : \mathbf{A}^2 \to \mathbf{A}^1$ be the map defined by $(s, t) \mapsto t$. η is a morphism.

Proof. Proof is similar to Example 5.82. ☐

Lemma 5.84. *Let $f(x, y) = y^2 - x^3$. $Z(y^2 - x^3)$ is an affine variety.*

Proof. We have to show that $f(x, y) = y^2 - x^3$ is irreducible. Assume $y^2 - x^3$ is not irreducible. Then

$$\begin{aligned} y^2 - x^3 &= (ax + by + c)(dx^2 + exy + gx + hy + jy^2 + k) \quad a, b, c, d, e, g, h, j, k \in K \\ &= adx^3 + (ae + bd)x^2y + (aj + be)xy^2 + (ah + bg + ce)xy + (ak + cg)x \\ &\quad + (bk + ch)y + (ag + cd)x^2 + (bh + cj)y^2 + bjy^3 + ck \end{aligned}$$

Hence we obtain the following equations.

$$\begin{array}{llll} ad = -1 & ae + bd = 0 & aj + be = 0 & ah + bg + ce = 0 \\ ak + cg = 0 & bk + ch = 0 & ag + cd = 0 & bh + cj = 1 \\ bj = 0 & ck = 0 & & \end{array}$$

Since $ck = 0$ either $c = 0$ or $k = 0$.
If $c = 0$ then we obtain the following equations.

$$\begin{array}{llll} ad = -1 & ae + bd = 0 & aj + be = 0 & ah + bg = 0 \\ ak = 0 & bk = 0 & ag = 0 & bh = 1 \\ bj = 0 & & & \end{array}$$

Using $ag = 0$ either $a = 0$ or $g = 0$. $a \neq 0$ because $ad = -1$. Therefore $g = 0$. Then we are left with the following.

$$\begin{array}{llll} ad = -1 & ae + bd = 0 & aj + be = 0 & ah = 0 \\ ak = 0 & bk = 0 & bh = 1 & \\ bj = 0 & & & \end{array}$$

Now we see that $ah = 0$ means either $a = 0$ or $h = 0$. But $a \neq 0$ since $ad = -1$ and $h \neq 0$ because $bh = 1$. Hence our initial assumption that $c = 0$ is wrong. Therefore we must assume that $k = 0$. If $k = 0$ we obtain

$$\begin{array}{llll} ad = -1 & ae + bd = 0 & aj + be = 0 & ah + bg + ce = 0 \\ cg = 0 & ch = 0 & ag + cd = 0 & bh + cj = 1 \\ bj = 0 & & & \end{array}$$

Using $cg = 0$ either $c = 0$ or $g = 0$.
If $c = 0$ then we obtain

$$\begin{array}{llll} ad = -1 & ae + bd = 0 & aj + be = 0 & ah + bg = 0 \\ & ch = 0 & ag = 0 & bh = 1 \\ bj = 0 & & & \end{array}$$

We see from $ag = 0$ that $a = 0$ or $g = 0$. $a \neq 0$ since $ad = -1$. Therefore $g = 0$. Hence we get

$$\begin{array}{llll} ad = -1 & ae + bd = 0 & aj + be = 0 & ah = 0 \\ & ch = 0 & bh = 1 & \\ bj = 0 & & & \end{array}$$

Now we see from $ah = 0$ that $a = 0$ or $h = 0$. $a \neq 0$ since $ad = -1$ and $h \neq 0$ since $bh = 1$. Therefore our assumption that $k = 0$ is wrong.
Hence we have shown that $y^2 - x^3$ is irreducible. From Theorem 2.55 we have that $Z(y^2 - x^3)$ is irreducible hence an affine variety. $\square$

Theorem 5.85. *Let $\varphi : \mathbf{A}^1 \to Z(y^2 - x^3)$ be defined by $t \mapsto (t^2, t^3)$. φ is a bijective bicontinuous morphism.*

Proof. First we show φ is a map. This means showing Im $(\varphi) \subset Z(y^2 - x^3)$. $\text{Im}(\varphi) = \{\varphi(t) \in \mathbf{A}^2 | t \in \mathbf{A}^1\} = \{(t^2, t^3) \in \mathbf{A}^2 | t \in \mathbf{A}^1\}$. Let $P \in$ Im(φ). Then $P = (t^2, t^3)$ for some $t \in \mathbf{A}^1$. $(t^3)^2 - (t^2)^3 = t^6 - t^6 = 0$. Hence $P \in Z(y^2 - x^3)$. Therefore Im $(\varphi) \subset Z(y^2 - x^3)$.

We will now show that $\phi : Z(y^2 - x^3) \to \mathbf{A}^1$ defined as,

$$(x, y) \mapsto \begin{cases} y/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is the inverse map of φ.

Let $(x, y) \in Z(y^2 - x^3)$.
Case 1: If $x = 0$ then $y = 0$. Hence we have

$$\begin{aligned}(\varphi \circ \phi)(0, 0) &= \varphi(\phi(0, 0)) \\ &= \varphi(0) \\ &= (0^2, 0^3) \\ &= (0, 0)\end{aligned}$$

Case 2: If $x \neq 0$ then

$$\begin{aligned}(\varphi \circ \phi)(x, y) &= \varphi(\phi(x, y)) \\ &= \varphi(\frac{y}{x}) \\ &= ((\frac{y}{x})^2, (\frac{y}{x})^3) \\ &= (\frac{y^2}{x^2}, \frac{y^3}{x^3}) \\ &= (\frac{x^3}{x^2}, \frac{y^3}{y^2}) \\ &= (x, y)\end{aligned}$$

Hence $\varphi \circ \phi$ is the identity map in $Z(y^2 - x^3)$.

Let $t \in \mathbf{A}^1$. $(\phi \circ \varphi)(t) = \phi(\varphi(t)) = \phi(t^2, t^3)$. If $t^2 \neq 0$ then $t \neq 0$. Hence we get $\phi(t^2, t^3) = \frac{t^3}{t^2} = t$. If $t^2 = 0$ then $t = 0$ hence $t^3 = 0$. Therefore we have $\phi(t^2, t^3) = \phi(0, 0) = 0$. Hence $(\phi \circ \varphi)$ is the identity in $\mathbf{A}^1$.

Hence ϕ is the inverse map of φ. This proves that φ is a bijective map.

We now show $\phi : Z(y^2 - x^3) \to \mathbf{A}^1$ is continuous.

φ^{-1} is continuous if and only if $(\varphi^{-1})^{-1}(D)$(inverse image in $Z(y^2 - x^3)$) is closed for all closed subsets $D \subseteq \mathbf{A}^1$. Let $D \subseteq \mathbf{A}^1$ be a closed subset. We would like to show $(\varphi^{-1})^{-1}(D) = \{(x, y) \in Z(y^2 - x^3) | \varphi^{-1}(x, y) \in D\}$ is closed in $Z(y^2 - x^3)$. The closed subsets in $\mathbf{A}^1$ are algebraic sets which are the finite sets,$\emptyset$ and $\mathbf{A}^1$ itself (Zariski Topology). $\varphi^{-1}(\emptyset) = \emptyset$. $\varphi^{-1}(\mathbf{A}^1) = Z(y^2 - x^3)$. $\emptyset$ and $Z(y^2 - x^3)$ are closed in $Z(y^2 - x^3)$. Therefore we need to show it is true for the finite sets. But φ is bijective. Hence the finite sets in $\mathbf{A}^1$ are mapped to by finite sets of points in $Z(y^2 - x^3)$. Finite sets of points in $Z(y^2 - x^3)$ are closed. Hence φ^{-1} is continuous.

Now we show φ is a morphism.

Let $t \in \mathbf{A}^1$ and x the coordinate function on $Z(y^2 - x^3)$. $(x \circ \varphi)(t) = t$. We have a map

$$\begin{aligned} x \circ \varphi : &\mathbf{A}^1 \to K \\ t &\mapsto t \end{aligned}$$

We need for each $t \in \mathbf{A}^1$ an open $U \subseteq \mathbf{A}^1$ with $t \in U$ and $g, h \in K[r]$ with $h(r) \neq 0$ on U and $x \circ \varphi = g(r)/h(r)$. We choose $U = \mathbf{A}^1$ and $x \circ \varphi = g(r)/h(r)$ where $g(r) = r$ and $h(r) = 1$. Hence we see $x \circ \varphi$ is regular.

Let $t \in \mathbf{A}^1$ and y the coordinate function on $Z(y^2 - x^3)$. $(y \circ \varphi)(t) = t^2$. We have a map

$$\begin{aligned} y \circ \varphi : &\mathbf{A}^1 \to K \\ t &\mapsto t^2 \end{aligned}$$

We choose our open set to be $\mathbf{A}^1$. $y \circ \varphi = g(r)/h(r)$ where $g(r) = r^2$ and $h(r) = 1$. Hence we see $y \circ \varphi$ is regular.

By Lemma 5.71 φ is a morphism.

φ is a bijective bicontinuous morphism.

□

Lemma 5.86. *$\varphi : \mathbf{A}^1 \to Z(y^2 - x^3)$ defined by $t \mapsto (t^2, t^3)$ is not an isomorphism.*

Proof. Since φ is bijective there is an inverse map $\varphi^{-1} : Z(y^2 - x^3) \to \mathbf{A}^1$ defined as, $(x, y) \mapsto$

$$\begin{cases} \frac{y}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We will show φ^{-1} is not a morphism.

φ^{-1} is continuous. Can we find a regular function f, such that $f \circ \varphi^{-1}$ is not regular for all points ? We choose the regular function defined by $f : t \mapsto t$. Assume $f \circ \varphi^{-1} : Z(y^2 - x^3) \to \mathbf{A}^1 \to K$ is regular at $P = (0,0)$. Then there exists an open $U \subset Z(y^2 - x^3)$ with $P \in U$ and $g, h \in K[x, y]$ such that $f \circ \varphi^{-1} = g/h$ on U and h is nowhere 0 on U. In particular $y/x = g/h$ on $U \backslash P$. Hence $hy = gx$ on $U \backslash P$. $U \backslash P$ is a non-empty open subset of an irreducible topological space hence it is dense. Therefore $hy = gx$ on $Z(y^2 - x^3)$. Applying φ to both sides we have $hy \circ \varphi = gx \circ \varphi$ on $\mathbf{A}^1$. Therefore

$$h(t^2, t^3)t^3 = g(t^2, t^3)t^2$$
$$h(t^2, t^3)t = g(t^2, t^3)$$

Taking $g(x, y) = a + bx + cy + dx^2 + \ldots$ then $g(t^2, t^3) = a + bt^2 + ct^3 + dt^4 + \ldots$. Since $g(t^2, t^3) = h(t^2, t^3)t$, the lowest possible term of g is bt^2 hence $a = 0$. Therefore $g \in t^2 K[t]$. Now $h(t^2, t^3) = bt + ct^2 + dt^3 + \ldots$. Hence $h(t^2, t^3)$ must have lowest term bt and hence $h(t^2, t^3) \in tK[t]$. Hence for $P = (0,0)$ we get $h(P) = 0$. This is a contradiction that $f \circ \varphi^{-1}$ is regular at $(0,0)$. Therefore φ^{-1} is not a morphism, and hence $\varphi : \mathbf{A}^1 \to Z(y^2 - x^3)$ defined by $t \mapsto (t^2, t^3)$ is not an isomorphism.

$\square$

6 Conics in $\mathbf{A}^2$ and $\mathbf{P}^2$

Lemma 6.1. *Let* $Y = Z(xy-1) \subseteq \mathbf{A}^2$. $Y \cong \mathbf{A}^1 \backslash \{0\}$.

Proof. First we define maps between Y and $\mathbf{A}^1\backslash\{0\}$. Define

$$\begin{aligned} \phi : \mathbf{A}^1\backslash\{0\} &\to Z(xy-1) \\ t &\mapsto (t, 1/t) \end{aligned}$$

and

$$\begin{aligned} \psi : Z(xy-1) &\to \mathbf{A}^1\backslash\{0\} \\ (x,y) &\mapsto x \end{aligned}$$

Let $t \in \mathbf{A}^1\backslash\{0\}$ and $x, y : \mathbf{P}^2 \to K$ coordinate functions. Then $(x \circ \phi)(t) = x(\phi(t)) = t$.

$$\begin{aligned} (x \circ \phi) : \mathbf{A}^1\backslash\{0\} &\to K \\ t &\mapsto t \end{aligned}$$

To show $x \circ \phi$ is regular on $\mathbf{A}^1\backslash\{0\}$ we need for each $t \in \mathbf{A}^1\backslash\{0\}$ an open $U \subseteq \mathbf{A}^1\backslash\{0\}$ with $t \in U$ and polynomials $g(r), h(r)$ with $h(r) \neq 0$ on U and $x \circ \phi = g(r)/h(r)$. We can choose $U = \mathbf{A}^1\backslash\{0\}$. $\mathbf{A}^1\backslash\{0\} \subseteq \mathbf{A}^1\backslash\{0\}$ is open. $t \in \mathbf{A}^1\backslash\{0\}$ and $x \circ \phi = g(r)/h(r)$ where $g(r) = t$ and $h(r) = 1$. Hence $x \circ \phi$ is regular on $\mathbf{A}^1\backslash\{0\}$.
Let $t \in \mathbf{A}^1 - \{0\}$. $(y \circ \phi)(t) = y(\phi(t)) = 1/t$.

$$\begin{aligned} (y \circ \phi) : \mathbf{A}^1\backslash\{0\} &\to K \\ t &\mapsto 1/t \end{aligned}$$

To show $y \circ \phi$ is regular on $\mathbf{A}^1\backslash\{0\}$ we need for each $t \in \mathbf{A}^1\backslash\{0\}$ an open $V \subseteq \mathbf{A}^1\backslash\{0\}$ with $t \in V$ and polynomials $g'(r), h'(r)$ such that $h'(r) \neq 0$ on V and $y \circ \phi = g'(r)/h'(r)$. We again choose $V = \mathbf{A}^1 - \{0\}$. $\mathbf{A}^1 - \{0\} \subseteq \mathbf{A}^1 - \{0\}$ is open. $t \in \mathbf{A}^1 - \{0\}$ and $y \circ \phi = g'(r)/h'(r)$ where $g'(r) = 1$ and $h'(r) = t$ for $t \neq 0$. Hence $y \circ \phi$ is regular on $\mathbf{A}^1 - \{0\}$. Therefore by Lemma 5.71 ϕ is a morphism.

Lemma 5.23 tells us ψ is a morphism if $j \circ \psi : Z(xy-1) \to \mathbf{A}^1$ is a morphism where $j : \mathbf{A}^1\backslash\{0\} \to \mathbf{A}^1$ is the inclusion map. $j \circ \psi : Z(xy-1) \to \mathbf{A}^1$ is the map

$$\begin{aligned} j \circ \psi : &Z(xy-1) \to \mathbf{A}^1 \\ &(x_P, y_P) \mapsto x_P \end{aligned}$$

Let $Q = (x_P, y_P) \in Z(xy-1)$ and $x, y : \mathbf{A}^2$ the coordinate functions. Then $(x \circ j \circ \psi)(Q) = (x \circ j)(\psi(Q)) = (x \circ j)(x_P) = x_P$.

$$\begin{aligned}(x \circ j \circ \psi) : Z(xy-1) &\to K \\ (x_P, y_P) &\mapsto x_P\end{aligned}$$

To show $x \circ j \circ \psi$ is regular on $Z(xy-1)$ we need an open $V \subseteq Z(xy-1)$ with $(x_P, y_P) \in V$ and $x \circ j \circ \psi = g(x,y)/h(x,y)$ with $h'(x,y) \neq 0$ on V. We choose $V = Z(xy-1)$. $Z(xy-1) \subseteq Z(xy-1)$ is open. $(x_P, y_P) \in Z(xy-1)$. $x \circ j \circ \psi = g(x,y)/h(x,y)$ where $g(x,y) = x$ and $h(x,y) = 1$. Therefore by Lemma 5.71 $j \circ \psi$ is a morphism. Hence by Lemma 5.23 ψ is a morphism.

Let $t \in \mathbf{A}^1 - \{0\}$.

$$\begin{aligned}(\psi \circ \phi)t &= \psi(\phi(t)) \\ &= \psi(t, 1/t) \\ &= t\end{aligned}$$

Hence $\psi \circ \phi$ is the identity map on $\mathbf{A}^1 - \{0\}$
Let $(x_P, y_P) \in Z(xy-1)$.

$$\begin{aligned}(\phi \circ \psi)(x_P, y_P) &= \phi(\psi(x_P, y_P)) \\ &= \phi(x_P) \\ &= (x_P, 1/x_P) \\ &= (x_P, y_P)\end{aligned}$$

Hence $\phi \circ \psi$ is the identity map on $Z(xy-1)$.
Therefore $Y \cong \mathbf{A}^1 - \{0\}$ □

Theorem 6.2. *Let $a, b, c, d, p, q \in K$. Let $f \in K[x,y]$ and $f' \in K[x', y']$. Let*

$$\begin{aligned}\phi : \mathbf{A}^2 &\to \mathbf{A}^2 \\ (x_P, y_P) &\mapsto x'_P = ax_P + by_P + p,\ y'_P = cx_P + dy_P + q\end{aligned}$$

Assume $f = f' \circ \phi$. Then,

(i) ϕ is a morphism.
(ii) $\phi(Z(f)) \subseteq Z(f')$
(iii) If $ad - bc \neq 0$ then ϕ is an isomorphism.
(iv) Assume $ad - bc \neq 0$. If f is irreducible, then f' is irreducible and $\phi_{|Z(f)} : Z(f) \to Z(f')$ is an isomorphism.

Proof. (i). By Lemma 5.71, ϕ is a morphism if and only if $x' \circ \phi$ and $y' \circ \phi$ are regular on $\mathbf{A}^2$ where x', y' are the coordinate functions. Let $Q = (x_P, y_P) \in \mathbf{A}^2$. Then

$$\begin{aligned} \phi(Q) &= (ax_P + by_P + p, cx + dy + q) \\ (x' \circ \phi)(Q) &= ax_P + by_P + p \end{aligned}$$

Hence we have a map

$$\begin{aligned} x' \circ \phi : \mathbf{A}^2 &\to K \\ (x_P, y_P) &\mapsto ax_P + by_P + p \end{aligned}$$

$\mathbf{A}^2 \subseteq \mathbf{A}^2$ is open, $Q \in \mathbf{A}^2$ and $(x' \circ \phi) = \frac{g(x,y)}{h(x,y)}$ where $g(x, y) = ax + by + p$ and $h(x, y) = 1$. Hence $x' \circ \phi = g/h$. Therefore $x' \circ \phi$ is regular on $\mathbf{A}^2$. Similarly

$$(y' \circ \phi)(Q) = cx_P + dy_P + q$$

Hence we have a map

$$\begin{aligned} y' \circ \phi : \mathbf{A}^2 &\to K \\ (x_P, y_P) &\mapsto cx_P + dy_P + q \end{aligned}$$

$\mathbf{A}^2 \subseteq \mathbf{A}^2$ is open, $(Q) \in \mathbf{A}^2$ and $(y' \circ \phi) = \frac{g'(x,y)}{h'(x,y)}$. where $g'(x, y) = cx + dy + q$ and $h'(x, y) = 1$. Hence $y' \circ \phi = g'/h'$. Therefore $y' \circ \phi$ is regular on $\mathbf{A}^2$. Therefore ϕ is a morphism.
(ii) Let $Q = (x_P, y_P) \in Z(f)$. Then

$$\begin{aligned} f(x_P, y_P) &= 0 \\ (f' \circ \phi)(x_P, y_P) &= 0 \\ f'(\phi(x_P, y_P)) &= 0 \end{aligned}$$

Hence $\phi(x_P, y_P) \in Z(f')$. Therefore $\phi(Z(f)) \subseteq Z(f')$
(iii). For this proof we change our map ϕ to matrix form. Hence we have,

$$\begin{aligned} \phi : \mathbf{A}^2 &\to \mathbf{A}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} \end{aligned}$$

Since $ad - bc \neq 0$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. [SLLA, Theorem 5.3 page 148.] Hence we obtain a map,

$$\psi : \mathbf{A}^2 \to \mathbf{A}^2$$
$$\begin{pmatrix} x' \\ y' \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} p \\ q \end{pmatrix}$$

By (i), ψ is a morphism.
Let $(x_P, y_P) \in \mathbf{A}^2$. Then,

$$\begin{aligned}
(\psi \circ \phi)(x_P, y_P) &= \psi(\phi(x_P, y_P) \\
&= \psi(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_P \\ y_P \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}) \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_P \\ y_P \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} p \\ q \end{pmatrix} \\
&= \begin{pmatrix} x_P \\ y_P \end{pmatrix}
\end{aligned}$$

Let $(x'_P, y'_P) \in \mathbf{A}^2$. Then

$$\begin{aligned}
(\phi \circ \psi)(x'_P, y'_P) &= \phi(\psi(x'_P, y'_P) \\
&= \phi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} x'_P \\ y'_P \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} p \\ q \end{pmatrix}) \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} x'_P \\ y'_P \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} \\
&= \begin{pmatrix} x'_P \\ y'_P \end{pmatrix}
\end{aligned}$$

Hence ϕ is an isomorphism.
(iv) We are given $ad - bc \neq 0$ and f is irreducible. Assume f' is not irreducible. Then $f'(x', y') = g'(x', y')h'(x', y')$ for $g', h' \in K[x', y']$ both not constant. We are also given $f = f' \circ \phi$. Let $(x_P, y_P) \in \mathbf{A}^2$.

$$\begin{aligned}
f(x_P, y_P) &= f'(\phi(x_P, y_P)) \\
&= f'(ax_P + by_P + p, cx_P + dy_P + q)
\end{aligned}$$

Hence

$$\begin{aligned}
f(x, y) &= f'(ax + by + p, cx + dy + q) \\
&= g'(ax + by + p, cx + dy + q)h'(ax + by + p, cx + dy + q)
\end{aligned}$$

for $g'(ax+by+p,cx+dy+q), h'(ax+by+p,cx+dy+q) \in K[x,y]$. We are given f is irreducible. Therefore, either $g'(ax+by+p,cx+dy+q)$ or $h'(ax+by+p,cx+dy+q)$ is constant.

Assume $g'(ax+by+p,cx+dy+q) = t$ for $t \in K$ constant. Therefore $g'(\phi(x_P,y_P)) = t$ for all $(x_P,y_P) \in \mathbf{A}^2$. Let $(x'_P,y'_P) \in \mathbf{A}^2$ be given. Since ϕ is surjective, $(x'_P,y'_P) = \phi(x_Q,y_Q)$ for some point $(x_Q,y_Q) \in \mathbf{A}^2$. Hence $g'(x'_P,y'_P) = g'(\phi(x_Q,y_Q)) = t$. Hence $g'(x',y') = t$. Therefore $g'(x',y')$ is constant. By assumption, $g'(x',y')$ was non-constant. If we assume $h'(ax+by+p,cx+dy+q) = s$ for some $s \in K$ constant the same argument results in $h'(x',y')$ being constant which again is a contradiction. Hence f' must be irreducible.

By (iii) ϕ is an isomorphism. This means we have an inverse morphism ψ such that $\phi \circ \psi = Id_{\mathbf{A}^2_{(x',y')}}$ and $\psi \circ \phi = Id_{\mathbf{A}^2_{(x,y)}}$. We are given $f = f' \circ \phi$. Hence $f \circ \psi = f' \circ \phi \circ \psi$. Therefore $f' = f \circ \psi$.

We have f, f' irreducible hence $Z(f)$ and $Z(f')$ are varieties. By (i) and (ii) ϕ is a morphism with $\phi(Z(f)) \subseteq Z(f')$. Hence we can restrict the map ϕ to $\phi_{|Z(f)} : Z(f) \to Z(f')$. Since $Z(f)$ and $Z(f')$ are varieties, $\phi_{|Z(f)}$ is a morphism by Lemma 5.74 and Lemma 5.24. ϕ is an isomorphism with inverse morphism ψ. Let $(x'_P,y'_P) \in Z(f')$. Then $f'(x'_P,y'_P) = 0$. Hence $(f \circ \psi)(x'_P,y'_P) = 0$. Therefore $f(\psi(x'_P,y'_P)) = 0$.Hence $\psi(x'_P,y'_P) \in Z(f')$. Therefore $\psi(Z(f)) \subseteq Z(f')$. We can restrict the map ψ to $\psi_{|Z(f')} : Z(f') \to Z(f)$. $\psi_{|Z(f')}$ is a morphism by Lemma 5.74 and Lemma 5.24. Let $(x'_P,y'_P) \in Z(f')$. Then

$$\begin{aligned}
(\phi_{|Z(f)} \circ \psi_{|Z(f')})(x'_P,y'_P) &= \phi_{|Z(f)}(\psi_{|Z(f')}(x'_P,y'_P)) \\
&= \phi_{|Z(f)}(\psi(x'_P,y'_P)) \\
&= \phi(\psi(x'_P,y'_P)) \\
&= (\phi \circ \psi)(x'_P,y'_P) \\
&= (x'_P,y'_P)
\end{aligned}$$

Hence $\phi_{|Z(f)} \circ \psi_{|Z(f')}$ is the identity map on $Z(f')$. Let $(x_P,y_P) \in Z(f)$. Then

$$\begin{aligned}
(\psi_{|Z(f')} \circ \phi_{|Z(f)})(x_P,y_P) &= \psi_{|Z(f')}(\phi_{|Z(f)}(x_P,y_P)) \\
&= \psi_{|Z(f')}(\phi(x_P,y_P)) \\
&= \psi(\phi(x_P,y_P)) \\
&= (\psi \circ \phi)(x_P,y_P) \\
&= (x_P,y_P)
\end{aligned}$$

Hence $\psi_{|Z(f')} \circ \phi_{|Z(f)}$ is the identity map on $Z(f)$. Hence $\phi_{|Z(f)} : Z(f) \to Z(f')$ is an isomorphism. □

Theorem 6.3. *Any conic in* $\mathbf{A}^2$ *is isomorphic to either* $\mathbf{A}^1$ *or* $\mathbf{A}^1 - \{0\}$

Proof. By definition a conic in $\mathbf{A}^2$ is the zero set of an irreducible polynomial of total degree 2. Let f be an irreducible polynomial of total degree 2. $f(x,y) = ax^2 + bxy + cy^2 + dx + ey + h$ for some $a,b,c,d,e,h \in K$ with a,b,c not all 0.

$$f(x,y) = ax^2 + bxy + cy^2 + dx + ey + h$$

Let $g(z) \in K[z]$ be the polynomial az^2+bz+c. K is algebraically closed. Hence we can factor $az^2 + bz + c$ and obtain,

$$\begin{aligned} az^2 + bz + c &= (rz+s)(tz+w) \quad r,s,t,w \in K \\ &\text{with } r,s \text{ not both 0 and } t,w \text{ not both 0.} \\ &= trz^2 + rzw + stz + sw \\ &= trz^2 + (rw+st)z + sw \end{aligned}$$

Hence $a = tr, b = rw + st, c = sw$. Substituting these equalities into $f(x,y)$ we get,

$$\begin{aligned} f(x,y) &= trx^2 + (rw+st)xy + swy^2 + dx + ey + h \\ &= trx^2 + rwxy + stxy + swy^2 + dx + ey + h \\ &= rx(tx+wy) + sy(tx+wy) + dx + ey + h \\ &= (tx+wy)(rx+sy) + dx + ey + h, \end{aligned}$$

Case 1. If $rx + sy$ and $tx + wy$ are linearly dependent then $i(rx + sy) + j(tx + wy) = 0$ for some $i, j \in K$, with i and j not both 0.[SLLA, page 10.]. Let $i = 0$. Then $j(tx+wy) = 0$ Hence $tx+wy$ is the zero vector. This is not allowed. Let $j = 0$. Then $rx + sy$ is the zero vector. This is not allowed. Hence $i \neq 0 \neq j$. Therefore we have $tx + wy = k(rx + sy)$ for some non zero $k \in K$, namely $k = -i/j$. Hence we have,

$$f(x,y) = k(rx+sy)^2 + dx + ey + h$$

Let $\overline{f}(x,y) = (rx+sy)^2 + \frac{d}{k}x + \frac{e}{k}y + \frac{h}{k}$. $\overline{f} = \frac{1}{k}f$. f is irreducible therefore $\overline{f}$ is irreducible. Put $rx + sy = x'$ and $dx/k + ey/k + h/k = -y'$. Then $\overline{f}(x,y)$ can be rewritten as a polynomial in $K[x',y']$ namely $f'(x',y') = x'^2 - y'$. We now have the following map,

$$\phi : \mathbf{A}^2_{x,y} \to \mathbf{A}^2_{x',y'}$$

$$(x_P, y_P) \mapsto (x'_P = rx_P + sy_P, y'_P = -\frac{dx_P}{k} - \frac{ey_P}{k} - \frac{h}{k})$$

By Theorem 6.2, part (i) ϕ is a morphism.
Let $(x_P, y_P) \in \mathbf{A}^2_{(x,y)}$. Then

$$\begin{aligned}
\phi(x_P, y_P) &= (rx_P + sy_P, -\frac{dx_P}{k} - \frac{ey_P}{k} - \frac{h}{k}) \\
f'(\phi(x_P, y_P)) &= (rx_P + sy_P)^2 + \frac{d}{k}x_P + \frac{e}{k}y_P + \frac{h}{k} \\
f'(\phi(x_P, y_P)) &= \overline{f}(x_P, y_P) \\
(f' \circ \phi)(x_P, y_P) &= \overline{f}(x_P, y_P)
\end{aligned}$$

Hence $\overline{f} = f' \circ \phi$. By Theorem 6.2 part (ii) $\phi(Z(\overline{f})) \subseteq Z(f')$

Assume $re - ds = 0$. Then $rx + sy$ and $dx + ey$ are linearly dependent. [SLLA, Theorem 4.2 page 159.] This means $dx + ey = \lambda(rx + sy)$ for some $\lambda \in K$. Hence we would have $k(rx + sy)^2 + \lambda(rx + sy) + h = 0$. Let $z = rx + sy$. Then our equation becomes $kz^2 + \lambda z + h = 0$. Since K is algebraically closed, $kz^2 + \lambda z + h = 0$ is reducible. Therefore f would be reducible which is not allowed. Hence $re - ds \neq 0$.Therefore $re/k - ds/k \neq 0$. By Theorem 6.2, part (iii), ϕ is an isomorphism.

We now have $re/k - ds/k \neq 0$. By Theorem 6.2 part (iv) since $\bar{f}$ is irreducible, f' is irreducible. By Theorem 6.2 part (iv)

$$\phi|_{Z(\overline{f})} : Z(\overline{f}) \to Z(f')$$

is an isomorphism. But $Z(\overline{f}) = Z(f)$. Hence $Z(f) \cong Z(f')$. i.e $Z(f) \cong Z(x' - y'^2)$. By example 5.79, $Z(x' - y'^2) \cong \mathbf{A}^1$. Hence $Z(f) \cong \mathbf{A}^1$.

Case 2. When $rx + sy$ and $tx + wy$ are linearly independent. This means $rw - st \neq 0$. $f(x, y) = (rx + sy)(tx + wy) + dx + ey + g$ for r, s not both 0 and t, w not both 0.

Let $x' = rx + sy$ and $y' = tx + wy$.

In matrix form we have

$$f(x, y) = x'y' + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + h$$

But

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r & s \\ t & w \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Therefore $f(x, y) \in K[x, y]$ can be expressed as a polynomial in $K[x', y']$ namely $f'(x', y') = x'y' + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} r & s \\ t & w \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} + h$

$\begin{pmatrix} r & s \\ t & w \end{pmatrix}^{-1} = \frac{1}{rw-st} \begin{pmatrix} w & -s \\ -t & r \end{pmatrix} = \begin{pmatrix} \frac{w}{rw-st} & \frac{-s}{rw-st} \\ \frac{-t}{rw-st} & \frac{r}{rw-st} \end{pmatrix}$
$\begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} \frac{w}{rw-st} & \frac{-s}{rw-st} \\ \frac{-t}{rw-st} & \frac{r}{rw-st} \end{pmatrix} = \begin{pmatrix} \frac{dw-et}{rw-st} & \frac{-ds+er}{rw-st} \end{pmatrix}$ Hence $f'(x', y') = x'y' + d'x' + e'y' + h$ with $d' = \frac{dw-et}{rw-st}$ and $e' = \frac{-ds+er}{rw-st}$.
Hence we have a map,

$$\begin{aligned} \Phi : \mathbf{A}^2 &\to \mathbf{A}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} r & s \\ t & w \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

We have $f = f' \circ \Phi$ by construction. We also have f irreducible hence f' is irreducible. Hence by Theorem 6.2 part (iv), Φ is a isomorphism.

$$\begin{aligned} f'(x', y') &= x'y' + d'x' + e'y' + h \\ &= x'y' + d'x' + e'y' + d'e' - d'e' + h \\ &= (x' + e')(y' + d') - d'e' + h \end{aligned}$$

Let $x'' = x' + e'$ and $y'' = y' + d'$. Then f' can be expressed as $f''(x'', y'') = x''y'' - h'$ where $h' = d'e' - h$. $h' \neq 0$. Otherwise f' is reducible and this is not true. Now we obtain the following map.

$$\begin{aligned} \Psi : \mathbf{A}^2 &\to \mathbf{A}^2 \\ \begin{pmatrix} x' \\ y' \end{pmatrix} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} e' \\ d' \end{pmatrix} \end{aligned}$$

We observe that $1 \cdot 1 - 0 \cdot 0 = 1$. We also have $f' = f'' \circ \Psi$ by construction. We know f' is irreducible hence f'' is irreducible. By Theorem 6.2 part (iv), Ψ is a isomorphism.
We can express f'' as $\overline{f''} = h'(\frac{x''}{h'}y'' - 1)$ and note here that $Z(f'') = Z(\overline{f''})$. Hence we have $h'(\frac{x''}{h'}y'' - 1) = 0$ from which we obtain $\frac{x''}{h'}y'' - 1 = 0$. With one final change of coordinates we let $x''' = \frac{x''}{h'}$ and $y''' = y''$. Hence $\overline{f''}$ can be expressed as $f''' = x'''y''' - 1$. We now obtain the map,

$$\begin{aligned} \omega : \mathbf{A}^2 &\to \mathbf{A}^2 \\ \begin{pmatrix} x'' \\ y'' \end{pmatrix} &\mapsto \begin{pmatrix} 1, 0 \\ 0, 1 \end{pmatrix} \begin{pmatrix} \frac{x''}{h'} \\ y'' \end{pmatrix} \end{aligned}$$

We observe that $1 \cdot 1 - 0 \cdot 0 = 1$. We also have $\overline{f''} = f''' \circ \omega$ by construction. We know $\bar{f''}$ irreducible hence f''' is irreducible. By Theorem 6.2 part (iv) ω

is an isomorphism. So we have $Z(f) \cong Z(f') \cong Z(f'') = Z(\overline{f''}) \cong Z(f''')$. Hence $Z(f) \cong Z(f''')$. So for case 2, $Z(f) \cong Z(xy-1)$. By Lemma 6.1, $Z(xy-1) \cong \mathbf{A}^1 - \{0\}$. Hence $Z(f) \cong \mathbf{A}^1 - \{0\}$.

Therefore any conic on $\mathbf{A}^2$ is isomorphic to either $\mathbf{A}^1$ or $\mathbf{A}^1 - \{0\}$ $\square$

Lemma 6.4. *Let X, Y be varieties, with $X = \bigcup_i U_i$ an open covering. Let $\phi : X \to Y$ be a map such that $\phi_{|U_i} : U_i \to Y$ is a morphism for each i. Then ϕ is a morphism.*

Proof. By Lemma 1.34 ϕ is continuous. Let $V \subseteq Y$ be an open subset and $f : V \to K$ a regular function. $\phi_{|U_i} : U_i \to Y$ is a morphism for each i hence we have that $f \circ \phi_{U|_i} : (\phi_{|U_i})^{-1}(V) \to K$ is regular. Let $P \in (\phi_{|U_i})^{-1}(V)$. We have that $P \in (\phi_{|U_j})^{-1}(V)$ for one j. We are given $f \circ \phi_{U_j} : (\phi_{|U_j})^{-1}(V) \to K$ is regular. Put $W = (\phi_{|U_j})^{-1}(V)$ Let $Q \in W$. Then

$$\begin{aligned}(f \circ \phi_{|U_j})(Q) &= f(\phi_{|U_j}(Q)) \\ &= f(\phi(Q)) \\ &= (f \circ \phi)(Q)\end{aligned}$$

$$(f \circ \phi)_{|W}(Q) = (f \circ \phi)(Q)$$

Hence $f \circ \phi_{|U_j}$ and $(f \circ \phi)_{|W}$ are the same map. Therefore $(f \circ \phi)_{|W}$ is regular on W. By Lemma 5.16 $f \circ \phi$ is regular on $\phi^{-1}(V)$. Hence ϕ is a morphism. $\square$

Lemma 6.5. *Let X be a variety. Let $Y \subseteq \mathbf{P}^n$ be an irreducible closed subset and $i : Y \to \mathbf{P}^n$ the inclusion map. Let $\phi : X \to Y$ be a map such that $i \circ \phi : X \to \mathbf{P}^n$ is a morphism. Then ϕ is a morphism.*

Proof. By Lemma 1.39 ϕ is continuous. Let $U \subseteq Y$ be open and $f : U \to K$ a regular function. Let $P \in \phi^{-1}(U)$. Then $\phi(P) \in U$. Since f is regular on U, there is an open neighbourhood $O \subseteq U$ with $\phi(P) \in O$ and homogeneous polynomials g, h of same degree, such that $h \neq 0$ on O and $f = g/h$ on O. By the subspace topology on Y, $O = W \cap Y$ for some open $W \subseteq \mathbf{P}^n$. Put $W' = W \cap (\mathbf{P}^n \backslash Z(h))$. Hence $W' \cap Y = O$. W' is open in $\mathbf{P}^n$ and $h \neq 0$ on W'. We have a regular function $\frac{g}{h} : W' \to K$. Put $\tilde{f} = g/h$. We have $i \circ \phi$ is a morphism, $W' \subseteq \mathbf{P}^n$ is open and $\tilde{f} : W' \to K$ is regular on W'. Hence

$\tilde{f} \circ i \circ \phi : (i \circ \phi)^{-1}(W') \to K$ is regular.

$$\begin{aligned}
(i \circ \phi)^{-1}(W') &= \{P \in X : (i \circ \phi)(P) \in W'\} \\
&= \{P \in X : i(\phi(P)) \in W'\} \\
&= \{P \in X : \phi(P) \in W' \cap Y\} \\
&= \{P \in X : \phi(P) \in O\} \\
&= \phi^{-1}(O)
\end{aligned}$$

Let $Q \in \phi^{-1}(O)$.

$$\begin{aligned}
(\tilde{f} \circ i \circ \phi)(Q) &= (\tilde{f} \circ i)(\phi(Q)) \\
&= \tilde{f}(i(\phi(Q)) \\
&= \tilde{f}(\phi(Q)) \\
&= f(\phi(Q)) \\
&= (f \circ \phi)(Q)
\end{aligned}$$

Hence $\tilde{f} \circ i \circ \phi$ and $f \circ \phi$ are the same map. Hence $f \circ \phi : \phi^{-1}(O) \to K$ is regular. By definition of regular function $f \circ \phi : \phi^{-1}(U) \to K$ is regular. Hence ϕ is a morphism. $\square$

From this point on, to distinguish which space we are in, we will denote zero sets in $\mathbf{A}^n$ by $Z_{\mathbf{A}^n}$ and zero sets in $\mathbf{P}^n$ by $Z_{\mathbf{P}^n}$.

We define the map,

$$\begin{aligned}
\tilde{\delta} : \mathbf{A}^2 \backslash \{0\} &\to Z_{\mathbf{P}^2}(y^2 - xz) \subseteq \mathbf{P}^2 \\
(s,t) &\mapsto [s^2 : ts : t^2]
\end{aligned}$$

Assume $s \neq 0$. Then $s^2 \neq 0$. Assume $t \neq 0$. Then $t^2 \neq 0$. Hence $[s^2 : ts : t^2] \in \mathbf{P}^2$. Also $(ts)^2 - s^2t^2 = t^2s^2 - s^2t^2 = 0$. Therefore $[s^2 : ts : t^2] \in Z_{\mathbf{P}^2}(y^2 - xz)$. Let $(s,t) \sim (s',t')$ in $\mathbf{A}^2 \backslash \{0\}$. Then $(s',t') = (\lambda s, \lambda t)$ for some non zero $\lambda \in K$. $\tilde{\delta}(s,t) = [s^2 : ts : t^2]$. $\tilde{\delta}(s',t') = \tilde{\delta}(\lambda s, \lambda t) = [\lambda^2 s^2 : \lambda t \lambda s : \lambda^2 t^2] = [s^2 : ts : t^2]$. Hence $\tilde{\delta}(s,t) = \tilde{\delta}(s',t')$. We can now construct the map $\delta : \mathbf{P}^1 \to Z_{\mathbf{P}^2}(y^2 - xz)$ by defining $\delta[s : t] = \tilde{\delta}(s,t)$.

$$\begin{array}{ccccccc}
(s,t) & \in & \mathbf{A}^2 \backslash \{0\} & \xrightarrow{\tilde{\delta}} & Z_{\mathbf{P}^2}(y^2 - xz) & \xrightarrow{i} & \mathbf{P}^2 \\
\downarrow & & \downarrow & \nearrow_{\delta} & & & \\
[s : t] & \in & \mathbf{P}^1 & & & &
\end{array}$$

Hence we have a well defined map $\delta : \mathbf{P}^1 \to Z_{\mathbf{P}^2}(y^2 - xz)$ which maps $[s : t] \mapsto [s^2 : ts : t^2]$. But we have the inclusion map $i : Z_{\mathbf{P}^2}(y^2 - xz) \hookrightarrow \mathbf{P}^2$ which maps $[s^2 : ts : t^2] \mapsto [s^2 : ts : t^2]$. Hence we have a well defined map $i \circ \delta : \mathbf{P}^1 \to \mathbf{P}^2$ which maps $[s : t] \mapsto [s^2 : ts : t^2]$.

Theorem 6.6. *$Z_{\mathbf{P}^2}(y^2 - xz) \subseteq \mathbf{P}^2$ is isomorphic to $\mathbf{P}^1$.*

Proof. We constructed a map $i \circ \delta : \mathbf{P}^1 \to \mathbf{P}^2$ defined by $[s : t] \mapsto [s^2 : ts : t^2]$. $\mathbf{P}^1$ is covered by the open sets $U_0^1 \cup U_1^1$. Recall $U_0^1 = \{P = [a_0 : a_1] \in \mathbf{P}^1 : a_0 \neq 0\}$ and $U_1^1 = \{Q = [b_0 : b_1] \in \mathbf{P}^1 : b_1 \neq 0\}$. We can restrict the map $i \circ \delta$ to the open set U_0^1. Let $[r : s] \in U_0^1$. Then $(i \circ \delta)[r : s] = i(\delta[r : s]) = i[r^2 : rs : s^2] = [r^2 : rs : s^2]$. The following diagram shows how we can construct a morphism $U_0^1 \to \mathbf{P}^2$.

$$
\begin{array}{ccccccc}
[a_0 : a_1] & \in & U_0^1 & & \mathbf{P}^2 & \ni & [1 : p : q] \\
\downarrow & & \phi_0^1 \downarrow & & \uparrow \psi_0^2 & & \uparrow \\
\frac{a_1}{a_0} & \in & \mathbf{A}^1 & \xrightarrow{\tau} & \mathbf{A}^2 & \ni & (p, q) \\
 & & \cup & & \cup & & \\
 & & t & \longmapsto & (t, t^2) & &
\end{array}
$$

By Lemma 5.44 $\phi_0^1 : U_0^1 \to \mathbf{A}^1$ is a morphism.
By Example 5.80 $\tau : \mathbf{A}^1 \to \mathbf{A}^2$ is a morphism.
By Lemma 5.44 $\psi_0^2 : \mathbf{A}^2 \to \mathbf{P}^2$ is a morphism.
$\psi_0^2 \circ \tau \circ \phi_0^1$ is a morphism by Lemma 5.19.

$$
\begin{aligned}
(\psi_0^2 \circ \tau \circ \phi_0^1)[r : s] =& (\psi_0^2 \circ \tau)(\phi_0^1[r : s]) \\
=& (\psi_0^2 \circ \tau)(\frac{s}{r}) \\
=& \psi_0^2(\tau(\frac{s}{r})) \\
=& \psi_0^2(\frac{s}{r}, \frac{s^2}{r^2}) \\
=& [1 : \frac{s}{r} : \frac{s^2}{r^2}] \\
=& [r^2 : rs : s^2]
\end{aligned}
$$

Hence $\psi_0^2 \circ \tau \circ \phi_0^1$ is equal to the restricted map $i \circ \delta : U_0^1 \to \mathbf{P}^2$ .Therefore the restricted map $i \circ \delta : U_0^1 \to \mathbf{P}^2$ is a morphism.

We restrict $i \circ \delta$ to the open set U_1^1. Let $[p : q] \in U_1^1$. Then $(i \circ \delta)[p : q] = i(\delta[p : q]) = i[p^2 : pq : q^2] = [p^2 : pq : q^2]$. The following diagram shows how we can construct a morphism $U_1^1 \to \mathbf{P}^2$.

$$\begin{array}{ccccccc}
[b_0 : b_1] & \in & U_1^1 & & \mathbf{P}^2 & \ni & [r : t : 1] \\
\downarrow & & \downarrow \phi_1^1 & & \uparrow \psi_2^2 & & \uparrow \\
\frac{b_0}{b_1} & \in & \mathbf{A}^1 & \xrightarrow{\gamma} & \mathbf{A}^2 & \ni & (r, t) \\
 & & \Cup & & \Cup & & \\
 & & s & \longmapsto & (s^2, s) & &
\end{array}$$

By Lemma 5.44 $\phi_1^1 : U_1^1 \to \mathbf{A}^1$ is a morphism.
By Example 5.80 $\gamma : \mathbf{A}^1 \to \mathbf{A}^2$ is a morphism.
By Lemma 5.44 $\psi_2^2 : \mathbf{A}^2 \to \mathbf{P}^2$ is a morphism.
$\psi_2^2 \circ \gamma \circ \phi_1^1$ is a morphism by Lemma 5.19.

$$\begin{aligned}
(\psi_2^2 \circ \gamma \circ \phi_1^1)[p : q] =& (\psi_2^2 \circ \gamma)(\phi_1^1[p : q]) \\
=& (\psi_2^2 \circ \gamma)(\frac{p}{q}) \\
=& \psi_2^2(\gamma(\frac{p}{q})) \\
=& \psi_2^2(\frac{p^2}{q^2}, \frac{p}{q}) \\
=& [\frac{p^2}{q^2} : \frac{p}{q} : 1] \\
=& [p^2 : pq : q^2]
\end{aligned}$$

Hence $\psi_2^2 \circ \gamma \circ \phi_1^1$ is equal to the restricted map $i \circ \delta : U_1^1 \to \mathbf{P}^2$. Therefore the restricted map $i \circ \delta : U_1^1 \to \mathbf{P}^2$ is a morphism.
By Lemma 6.4 $i \circ \delta$ is a morphism.
We have that $Z_{\mathbf{P}^2}(y^2 - xy)$ is a closed irreducible subset of $\mathbf{P}^2$ therefore $\delta : \mathbf{P}^1 \to Z_{\mathbf{P}^2}(y^2 - xz)$ is a morphism by Lemma 6.5.

We define ψ as follows,

$$\begin{aligned}
\psi : Z_{\mathbf{P}^2}(y^2 - xz) &\to \mathbf{P}^1 \\
[x : y : z] &\mapsto \begin{cases} [x : y] \text{ if } x \neq 0 \\ [y : z] \text{ if } z \neq 0 \end{cases}
\end{aligned}$$

We will show ψ is a well defined map by showing,

Condition 1 : Let $[x : y : z] \in Z_{\mathbf{P}^2}(y^2 - xz)$. Then $x \neq 0$ or $z \neq 0$.
Condition 2 : If $x \neq 0$ and $z \neq 0$ then $[x : y] = [y : z]$.
Condition 3 : If $[x : y : z] = [x' : y' : z']$ then $\psi[x : y : z] = \psi[x' : y' : z']$

Proof of 1: Let $[x : y : z] \in Z_{\mathbf{P}^2}(y^2 - xz) \subseteq \mathbf{P}^2$. Then $y^2 = xz$. Assume $x = 0 = z$. Then $y^2 = 0 \cdot 0$ which gives $y^2 = 0$. Hence $y = 0$. Therefore $x = y = z = 0$. Hence $[x : y : z] \notin \mathbf{P}^2$. Therefore x and z cannot both be 0.

Proof of 2: Let $x \neq 0$ and $z \neq 0$. $[x : y] = [y^2/z : y] = [y^2 : yz] = [y : z]$. Hence we see if both x and z are non zero then $[x : y] = [y : z]$.

Proof of 3: Let $[x, y, z] = [x' : y' : z']$. Then $(x', y', z') = (\lambda x, \lambda y, \lambda z)$.

If $x \neq 0$ then $\psi[x : y : z] = [x : y]$ and $\psi[x' : y' : z'] = \psi[\lambda x : \lambda y : \lambda z] = [\lambda x : \lambda y]$. $[x : y] = [x' : y']$ since $(x', y') = \lambda(x, y)$.

If $z \neq 0$ then $\psi[x : y : z] = [y : z]$ and $\psi[x' : y' : z'] = \psi[\lambda x : \lambda y : \lambda z] = [\lambda y : \lambda z]$. $[y : z] = [y' : z']$ since $(y', z') = \lambda(y, z)$.

Hence ψ is a well defined map.

We now show that $\psi \circ \delta$ and $\delta \circ \psi$ are the identity maps on $\mathbf{P}^1$ and $Z_{\mathbf{P}^2}(y^2 - xz)$ respectively.
Let $[s : t] \in \mathbf{P}^1$.

Case 1: $s \neq 0$. If $s \neq 0$ then $s^2 \neq 0$.

$$\begin{aligned}(\psi \circ \delta)[s : t] &= \psi(\delta[s : t]) \\ &= \psi[s^2 : ts : t^2] \\ &= [s^2 : ts] \\ &= [s : t]\end{aligned}$$

Case 2: $s = 0$. If $s = 0$ then $[s : t] = [0 : 1]$.

$$\begin{aligned}(\psi \circ \delta)[0 : 1] &= \psi(\delta[0 : 1]) \\ &= \psi[0^2 : 0 \cdot 1 : 1^2] \\ &= \psi[0 : 0 : 1] \\ &= [0 : 1]\end{aligned}$$

Hence $\psi \circ \delta$ is the identity map on $\mathbf{P}^1$.

Let $[x : y : z] \in Z_{\mathbf{P}^2}(y^2 - xz)$.
Case 1: $x \neq 0$.

$$\begin{aligned}(\delta \circ \psi)[x : y : z] &= \delta(\psi[x : y : z]) \\ &= \delta[x : y] \\ &= [x^2 : xy : y^2] \\ &= [x^2 : xy : xz] \\ &= [x : y : z]\end{aligned}$$

Case 2: $x = 0$. If $x = 0$ we have previously shown $z \neq 0$.

$$\begin{aligned}(\delta \circ \psi)[x : y : z] &= \delta(\psi[x : y : z]) \\ &= \delta[y : z] \\ &= [y^2 : yz : z^2] \\ &= [xz : yz : z^2] \\ &= [x : y : z]\end{aligned}$$

Hence $\delta \circ \psi$ is the identity map on $Z_{\mathbf{P}^2}(y^2 - xz)$. This shows δ is a bijective map.

We now show ψ is a morphism.

$Z_{\mathbf{P}^2}(y^2 - xz)$ is covered by the open sets $(U_0^2 \cap Z_{\mathbf{P}^2}(y^2 - xz)) \cup (U_2^2 \cap Z_{\mathbf{P}^2}(y^2 - xz))$. We showed this earlier in the Theorem when we proved Condition 1.
We restrict the map $\psi : Z_{\mathbf{P}^2}(y^2 - xz) \to \mathbf{P}^1$ to the open set $U_0^2 \cap Z_{\mathbf{P}^2}(y^2 - xz)$. Let $[x : y : z] \in U_0^2 \cap Z_{\mathbf{P}^2}(y^2 - xz)$. This means $x \neq 0$. Hence $\psi[x : y : z] = [x : y]$. The following diagram shows how we can construct a morphism from $U_0^2 \cap Z_{\mathbf{P}^2}(y^2 - xz)$ to P^1.

$$\begin{array}{ccccccc}
[x : y : z] & \in & U_0^2 \cap Z_{\mathbf{P}^2}(y^2 - xz) & & \mathbf{P}^1 & \ni & [1 : p] \\
\downarrow & & \downarrow i & & \uparrow \psi_0^1 & & \uparrow \\
[x : y : z] & \in & U_0^2 & & & & \\
\downarrow & & \downarrow \phi_0^2 & & & & \\
(\frac{y}{x}, \frac{z}{x}) & \in & \mathbf{A}^2 & \xrightarrow{\omega} & \mathbf{A}^1 & \ni & p \\
 & & \Cup & & \Cup & & \\
 & & (t, s) & \longmapsto & t & &
\end{array}$$

By Lemma 5.21 the inclusion map $i : U_0^2 \cap Z_{\mathbf{P}^2}(y^2 - xz) \to U_0^2$ is a morphism.
By Lemma 5.44 $\phi_0^2 : U_0^2 \to \mathbf{A}^2$ is a morphism.
$\phi_0^2 \circ i : U_0^2 \cap Z_{\mathbf{P}^2}(y^2 - xz) \to \mathbf{A}^2$ is a morphism by Lemma 5.19.
By Example 5.82 $\omega : A^2 \to \mathbf{A}^1$ is a morphism.
By Lemma 5.44 $\psi_0^1 : \mathbf{A}^1 \to \mathbf{P}^1$ is a morphism.

Lemma 5.19 gives us $\psi_0^1 \circ \omega \circ \phi_0^2 \circ i$ is a morphism.

$$
\begin{aligned}
(\psi_0^1 \circ \omega \circ \phi_0^2 \circ i)[x : y : z] &= (\psi_0^1 \circ \omega \circ \phi_0^2)(i[x : y : z]) \\
&= (\psi_0^1 \circ \omega \circ \phi_0^2)[x : y : z] \\
&= (\psi_0^1 \circ \omega)(\phi_0^2[x : y : z]) \\
&= (\psi_0^1 \circ \omega)(\frac{y}{x}, \frac{z}{x}) \\
&= \psi_0^1(\omega(\frac{y}{x}, \frac{z}{x})) \\
&= \psi_0^1(\frac{y}{x}) \\
&= [1 : \frac{y}{x}] \\
&= [x : y]
\end{aligned}
$$

Hence $\psi_0^1 \circ \omega \circ \phi_0^2 \circ i$ is equal to the restricted map $\psi : U_0^2 \cap Z_{\mathbf{P}^2}(y^2 - xz) \to P^1$. Therefore the restricted map $\psi : U_0^2 \cap Z_{\mathbf{P}^2}(y^2 - xz) \to \mathbf{P}^1$ is a morphism.

We restrict the map $\psi : Z_{\mathbf{P}^2}(y^2 - xz) \to \mathbf{P}^1$ to the open set $U_2^2 \cap Z_{\mathbf{P}^2}(y^2 - xz)$. Let $[x : y : z] \in U_2^2 \cap Z_{\mathbf{P}^2}(y^2 - xz)$. This means $z \neq 0$. $\psi[x : y : z] = [y : z]$.

$$
\begin{array}{ccccccc}
[x:y:z] & \in & U_2^2 \cap Z_{\mathbf{P}^2}(y^2-xz) & & \mathbf{P}^1 & \ni & [q:1] \\
\downarrow & & \downarrow i & & \uparrow \psi_1^1 & & \\
[x:y:z] & \in & U_2^2 & & & & \uparrow \\
\downarrow & & \downarrow \phi_2^2 & & & & \\
(\frac{x}{z},\frac{y}{z}) & \in & \mathbf{A}^2 & \xrightarrow{\eta} & \mathbf{A}^1 & \ni & q \\
 & & \cup & & \cup & & \\
 & & (r,p) & \longmapsto & p & &
\end{array}
$$

By Lemma 5.21 the inclusion map $i : U_2^2 \cap Z_{\mathbf{P}^2}(y^2 - xz) \to U_2^2$ is a morphism. By Lemma 5.44 $\phi_2^2 : U_2^2 \to \mathbf{A}^2$ is a morphism. Hence $\phi_2^2 \circ i : U_2^2 \cap Z_{\mathbf{P}^2}(y^2 - xz) \to \mathbf{A}^2$ is a morphism by Lemma 5.19.

By Example 5.83 $\eta : \mathbf{A}^2 \to \mathbf{A}^1$ is a morphism.

By Lemma 5.44 $\psi_1^1 : \mathbf{A}^1 \to \mathbf{P}^1$ is a morphism.

Lemma 5.19 gives us $\psi_1^1 \circ \eta \circ \phi_2^2 \circ i$ is a morphism.

$$\begin{aligned}\psi_1^1 \circ \eta \circ \phi_2^2 \circ i)[x:y:z] &= (\psi_1^1 \circ \eta \circ \phi_2^2)(i[x:y:z]) \\ &= (\psi_1^1 \circ \eta \circ \phi_2^2)[x:y:z] \\ &= (\psi_1^1 \circ \eta)(\phi_2^2[x:y:z]) \\ &= (\psi_1^1 \circ \eta)(\frac{x}{z}, \frac{y}{z}) \\ &= \psi_1^1(\eta(\frac{x}{z}, \frac{y}{z})) \\ &= \psi_1^1(\frac{y}{z}) \\ &= [\frac{y}{z} : 1] \\ &= [y:z]\end{aligned}$$

Hence $\psi_1^1 \circ \eta \circ \phi_2^2 \circ i$ is equal to the restricted map $\psi : U_2^2 \cap Z_{\mathbf{P}^2}(y^2 - xz) \to \mathbf{P}^1$.
Therefore the restricted map $\psi : U_2^2 \cap Z_{\mathbf{P}^2}(y^2 - xz) \to \mathbf{P}^1$ is a morphism.
By Lemma 6.4, $\psi : Z_{\mathbf{P}^2}(y^2 - xz) \to \mathbf{P}^1$ is a morphism.
Hence $\mathbf{P}^1 \cong Z_{\mathbf{P}^2}(y^2 - xz)$. □

Lemma 6.7. *Let π be the projection map*

$$\begin{aligned}\pi : \mathbf{A}^3 \backslash \{(0,0,0)\} &\to \mathbf{P}^2 \\ (x,y,z) &\mapsto [x:y:z]\end{aligned}$$

π is a morphism.

Proof. Let $V \subseteq \mathbf{P}^2$ be a closed subset. Closed subsets in $\mathbf{P}^2$ are zero sets of homogeneous polynomials. Hence $V = Z_{\mathbf{P}^2}(T) = \{Q \in \mathbf{P}^2 : f(Q) = 0 \text{ for each } f \in T\}$ where T is a set of homogeneous polynomials f.

$$\begin{aligned}\pi^{-1}(Z_{\mathbf{P}^2}(T)) &= \{P \in \mathbf{A}^3 \backslash \{(0,0,0)\} : \pi(P) \in Z_{\mathbf{P}^2}(T)\} \\ &= \{(x,y,z) \in \mathbf{A}^3 \backslash \{(0,0,0)\} : [x:y:z] \in Z_{\mathbf{P}^2}(T)\} \\ &= \{(x,y,z) \in \mathbf{A}^3 \backslash \{(0,0,0)\} : f_i(x,y,z) = 0\} \\ &= Z_{\mathbf{A}^3}(T) - \{(0,0,0)\}\end{aligned}$$

$Z_{\mathbf{A}^3}(T) - \{(0,0,0)\} = Z_{\mathbf{A}^3}(T) \cap \mathbf{A}^3 \backslash \{(0,0,0)\}$. Hence by the subspace topology on $\mathbf{A}^3 \backslash \{(0,0,0)$, $\pi^{-1}(V) = Z_{\mathbf{A}^3}(T) - \{(0,0,0)\}$ is closed in $\mathbf{A}^3 \backslash \{(0,0,0)$.
Therefore π is continuous.

Let $U \subseteq \mathbf{P}^2$ be an open subset and $f : U \to K$ regular. Note that since π is continuous, $\pi^{-1}(U)$ is open in $\mathbf{A}^3 \backslash \{(0,0,0)\}$. Let $Q \in \pi^{-1}(U)$. Then $\pi(Q) \in U$. Hence there exists an open neighbourhood $O \subset U$ with $\pi(Q) \in O$ and homogeneous polynomials g, h of same degree with $h \neq 0$ on O and $f = g/h$

on O. By Lemma 1.18 O is open in $\mathbf{P}^2$. Hence $\pi^{-1}(O)$ is open in $\mathbf{A}^3\backslash\{(0,0,0)\}$. By the subspace topology on $\pi^{-1}(U)$, $\pi^{-1}(O)$ is open in $\pi^{-1}(U)$. Also since $\pi(Q) \in O$ we have $Q \in \pi^{-1}(O)$. Hence we have found an open neighbourhood $\pi^{-1}(O) \subseteq \pi^{-1}(U)$ with $Q \in \pi^{-1}(O)$.
Let $P = (x, y, z) \in \pi^{-1}(O)$.

$$\begin{aligned}
(f \circ \pi)(x, y, z) &= f(\pi(x, y, z)) \\
&= \frac{g}{h}(\pi(x, y, z)) \\
&= \frac{g}{h}[x : y : z] \\
&= \frac{g(x, y, z)}{h(x, y, z)} \\
&= \frac{g(P)}{h(P)}
\end{aligned}$$

By Lemma 5.4 $f \circ \pi$ is regular.

Hence π is a morphism. ☐

Lemma 6.8. *Let $a, b, c, d, e, g, h, j, l \in K$. Let σ be the map,*

$$\begin{aligned}
\sigma : \mathbf{A}^2 &\to \mathbf{A}^3 \\
(p, q) &\mapsto (a + bp + cq, d + ep + gq, h + jp + lq)
\end{aligned}$$

σ is a morphism.

Proof. Let $(p, q) \in \mathbf{A}^2$ and x, y, z the coordinate functions on $\mathbf{A}^3$. Then $(x \circ \sigma)(p, q) = x(\sigma(p, q)) = a + bp + cq$.

$$\begin{aligned}
(x \circ \sigma) : \mathbf{A}^2 &\to K \\
(p, q) &\mapsto a + bp + cq
\end{aligned}$$

To show $x \circ \sigma$ is regular on $\mathbf{A}^2$ we need for each $Q \in \mathbf{A}^2$ an open $U \subseteq \mathbf{A}^2$ with $Q \in U$ and $x \circ \sigma = g(s, t)/h(s, t)$ with $h(s, t) \neq 0$ on U. We choose $U = \mathbf{A}^2$. $\mathbf{A}^2 \subseteq \mathbf{A}^2$ is open. $(Q) \in \mathbf{A}^2$ and $x \circ \sigma = g(s, t)/h(s, t)$ where $g(s, t) = a + bs + ct$ and $h(s, t) = 1$. Hence $x \circ \sigma$ is regular on $\mathbf{A}^2$.

Let $R = (p, q) \in \mathbf{A}^2$.
Then $(y \circ \sigma)(p, q) = y(\sigma(p, q)) = d + ep + gq$.

$$\begin{aligned}
(y \circ \sigma) : \mathbf{A}^2 &\to K \\
(p, q) &\mapsto d + ep + gq
\end{aligned}$$

To show $y \circ \sigma$ is regular on $\mathbf{A}^2$ we need for each $R \in \mathbf{A}^2$ an open $U \subseteq \mathbf{A}^2$ with $R \in U$ and $y \circ \sigma = g(s,t)/h(s,t)$ with $h(s,t) \neq 0$ on U. We choose $U = \mathbf{A}^2$. $\mathbf{A}^2 \subseteq \mathbf{A}^2$ is open. $R \in \mathbf{A}^2$ and $y \circ \sigma = g(s,t)/h(s,t)$ where $g(s,t) = d + es + gt$ and $h(s,t) = 1$. Hence $y \circ \sigma$ is regular on $\mathbf{A}^2$.

Let $T = (p,q) \in \mathbf{A}^2$.
Then $(z \circ \sigma)(T) = z(\sigma(T)) = h + jp + lq$.

$$\begin{aligned}(z \circ \sigma) : \mathbf{A}^2 &\to K \\ (p,q) &\mapsto h + jp + lq\end{aligned}$$

To show $z \circ \sigma$ is regular on $\mathbf{A}^2$ we need for each $T \in \mathbf{A}^2$ an open $U \subseteq \mathbf{A}^2$ with $T \in U$ and $z \circ \sigma = g(s,t)/h(s,t)$ with $h(s,t) \neq 0$ on U. We choose $U = \mathbf{A}^2$. $\mathbf{A}^2 \subseteq \mathbf{A}^2$ is open. $T \in \mathbf{A}^2$ and $z \circ \sigma = g(s,t)/h(s,t)$ where $g(s,t) = h + js + lt$ and $h(s,t) = 1$. Hence $z \circ \sigma$ is regular on $\mathbf{A}^2$.

Therefore by Lemma 5.71 σ is a morphism. □

Corollary 6.9. We note here that, consequently,

$$\begin{aligned}(i)\quad \sigma' : \mathbf{A}^2 &\to \mathbf{A}^3 \\ (p,q) &\mapsto (ap + b + cq, dp + e + gq, hp + j + lq) \\ &\text{and} \\ (ii)\quad \sigma'' : \mathbf{A}^2 &\to \mathbf{A}^3 \\ (p,q) &\mapsto (ap + bq + c, dp + eq + g, hp + jq + l)\end{aligned}$$

are both morphisms.

Lemma 6.10. *Let $a,b,c,d,e,g,h,j,l \in K$. Assume the determinant of* $\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}$ *is not zero. Let ω be defined as ,*

$$\begin{aligned}\omega : \mathbf{A}^2 &\to \mathbf{A}^3 \backslash \{(0,0,0)\} \\ (p,q) &\mapsto (a + bp + cq, d + ep + gq, h + jp + lq)\end{aligned}$$

Then,

(i) ω is a well defined map.

(ii) ω is a morphism.

Proof. (i). We only need to show that $(a + bp + cq, d + ep + gq, h + jp + lq) \in \mathbf{A}^3 \backslash \{(0,0,0)\}$. That is that $a + bp + cq, d + ep + gq, h + jp + lq$ are not all 0. Assume $a + bp + cq = d + ep + gq = h + jp + lq = 0$.

Then in matrix form we have

$$\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix} \begin{pmatrix} 1 \\ p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We are given that the determinant of $\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}$ is not zero. Hence $\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}$ is invertible. Therefore we obtain,

$$\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}^{-1} \begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix} \begin{pmatrix} 1 \\ p \\ q \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

But this would give $\begin{pmatrix} 1 \\ p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and clearly $1 \neq 0$. Hence our initial assumption is incorrect and $a + bp + cq, d + ep + gq, h + jp + lq$ are not all 0. Therefore $(a + bp + cq, d + ep + gq, h + jp + lq) \in \mathbf{A}^3 \backslash \{(0, 0, 0)\}$ and ω is well defined.
(ii) Let $i : \mathbf{A}^3 \backslash \{(0, 0, 0)\} \to \mathbf{A}^3$ be the inclusion map. By Lemma 5.23 ω is a morphism if $i \circ \omega : \mathbf{A}^2 \mapsto \mathbf{A}^3$ is a morphism. $i \circ \omega$ is the map

$$\mathbf{A}^2 \xrightarrow{\quad\omega\quad} \mathbf{A}^3 \backslash \{(0,0,0)\} \xrightarrow{\quad i \quad} \mathbf{A}^3$$

$$(p, q) \quad \mapsto \quad (a + bp + cq, d + ep + gq, h + jp + lq) \quad \mapsto \quad (a + bp + cq,$$
$$d + ep + gq,$$
$$h + jp + lq)$$

Hence we see $i \circ \omega : \mathbf{A}^2 \to \mathbf{A}^3$ is exactly the map in Lemma 6.8 which we have proved is a morphism. Hence ω is a morphism. □

Lemma 6.11. *Let $a, b, c, d, e, g, h, j, l \in K$. Assume det $\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}$ is not zero. Let ω' be defined as ,*

$$\omega' : \mathbf{A}^2 \to \mathbf{A}^3 \backslash \{(0, 0, 0)\}$$
$$(p, q) \mapsto (ap + b + cq, dp + e + gq, hp + j + lq)$$

Then,

(i) ω' is a well defined map.
(ii) ω' is a morphism.

Proof. (i). We only need to show that $(ap+b+cq, dp+e+gq, hp+jp+lq) \in \mathbf{A}^3\backslash\{(0,0,0)\}$. Assume $ap+b+cq = dp+e+gq = hp+j+lq = 0$.
Then in matrix form we have

$$\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix} \begin{pmatrix} p \\ 1 \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}$ is invertible by assumption. Therefore, using a similar calculation as in Lemma 6.10 we would obtain,
$\begin{pmatrix} p \\ 1 \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and clearly $1 \neq 0$. Hence our initial assumption is incorrect and $ap+b+cq, dp+e+gq, hp+j+lq$ are not all 0. Therefore $(ap+b+cq, dp+e+gq, hp+j+lq) \in \mathbf{A}^3\backslash\{(0,0,0)\}$ and ω' is well defined.
(ii) Let $i : \mathbf{A}^3\backslash\{(0,0,0)\} \to \mathbf{A}^3$ be the inclusion map. By Lemma 5.23 ω' is a morphism if $i \circ \omega' : \mathbf{A}^2 \mapsto \mathbf{A}^3$ is a morphism. $i \circ \omega'$ is the map

$$\begin{array}{ccccc} \mathbf{A}^2 & \xrightarrow{\omega'} & \mathbf{A}^3\backslash\{(0,0,0)\} & \xrightarrow{i} & \mathbf{A}^3 \\ (p,q) & \mapsto & (ap+b+cq, dp+e+gq, hp+j+lq) & \mapsto & (ap+b+cq, \\ & & & & dp+e+gq, \\ & & & & hp+j+lq) \end{array}$$

Hence we see $i \circ \omega' : \mathbf{A}^2 \to \mathbf{A}^3$ is exactly the map σ' in Corollary 6.9 (i) which is a morphism. Hence ω' is a morphism. ☐

Lemma 6.12. *Let $a, b, c, d, e, g, h, j, l \in K$. Assume det $\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}$ is not* *zero. Let ω'' be defined as ,*

$$\begin{aligned}\omega'' : \mathbf{A}^2 &\to \mathbf{A}^3 \backslash \{(0,0,0)\}\\ (p,q) &\mapsto (ap+bq+c, dp+eq+g, hp+jq+l)\end{aligned}$$

Then,

(i) ω'' is a well defined map.
(ii) ω'' is a morphism.

Proof. (i). We only need to show that $(ap+bq+c, dp+eq+g, hp+jq+l) \in \mathbf{A}^3 \backslash \{(0,0,0)\}$. Assume $ap+b+cq = dp+e+gq = hp+j+lq = 0$.
Then in matrix form we have

$$\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix} \begin{pmatrix} p \\ q \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}$ is invertible. Therefore, using a similar calculation as in Lemma 6.10 we would obtain ,
$\begin{pmatrix} p \\ q \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. and clearly $1 \neq 0$. Hence our initial assumption is incorrect and $ap+bq+c, dp+eq+g, hp+jq+l$ are not all 0. Therefore $(ap+bq+c, dp+eq+g, hp+jq+l) \in \mathbf{A}^3 \backslash \{(0,0,0)\}$ and ω'' is well defined.
(ii) Let $i : \mathbf{A}^3 \backslash \{(0,0,0)\} \to \mathbf{A}^3$ be the inclusion map. By Lemma 5.23 ω'' is a morphism if $i \circ \omega'' : \mathbf{A}^2 \mapsto \mathbf{A}^3$ is a morphism. $i \circ \omega''$ is the map

$$\begin{array}{ccccc} \mathbf{A}^2 & \xrightarrow{\omega'} & \mathbf{A}^3 \backslash \{(0,0,0)\} & \xrightarrow{i} & \mathbf{A}^3 \\ (p,q) & \mapsto & (ap+bq+c, dp+eq+g, hp+jq+l) & \mapsto & (ap+bq+c, \\ & & & & dp+eq+g, \\ & & & & hp+jq+l) \end{array}$$

Hence we see $i \circ \omega'' : \mathbf{A}^2 \to \mathbf{A}^3$ is exactly the map σ'' in Corollary 6.9 (ii) which is a morphism. Hence ω'' is a morphism. ☐

Theorem 6.13. *Let $a, b, c, d, e, g, h, j, l \in K$. Assume $det \begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}$ is non* *zero. Let*

$$\begin{aligned}\tilde{\tau} : \mathbf{A}^3 &\to \mathbf{A}^3\\ (x_P, y_P, z_P) &\mapsto (x'_P, y'_P, z'_P)\\ &\textit{with}\\ &x'_P = ax_P + by_P + cz_P\\ &y'_P = dx_P + ey_P + gz_P\\ &z_P = hx_P + jy_P + lz_P\end{aligned}$$

Let $f \in K[x, y, z]$ *and* $f' \in K[x', y', z']$*. Assume that* $f = f' \circ \tilde{\tau}$ *and assume* f, f' *are both homogeneous.*
Then,

(i) $\tilde{\tau}$ *induces a well defined map,*

$$\begin{aligned}\tau : \mathbf{P}^2 &\to \mathbf{P}^2\\ [x_P : y_P : z_P] &\mapsto [x'_P : y'_P : z'_P]\end{aligned}$$

(ii) τ *is a morphism.*
(iii) τ *is an isomorphism.*
(iv) $\tau(Z_{\mathbf{P}^2}(f)) \subseteq Z_{\mathbf{P}^2}(f'), Z_{\mathbf{P}^2}$ *is zero set in* $\mathbf{P}^2$*.*
(v) If f *is irreducible, then* f' *is irreducible and* $\tau_{|Z_{\mathbf{P}^2}(f)} : Z_{\mathbf{P}^2}(f) \to Z_{\mathbf{P}^2}(f')$ *is an isomorphism.*

Proof. (i): First we restrict $\tilde{\tau}$ to $\mathbf{A}^3\backslash\{(0,0,0)\}$. This gives a map $\mathbf{A}^3\backslash\{(0,0,0\} \to \mathbf{A}^3$. Let $(x_P, y_P, z_P) \in \mathbf{A}^3\backslash\{(0,0,0)\}$. In matrix form we have

$$\begin{pmatrix} x'_P \\ y'_P \\ z'_P \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix} \begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix}$$

Assume $x'_P = y'_P = z'_P = 0$. Then

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix} \begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix}$$

By assumption $\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}$ is invertible. Hence

$$\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}^{-1} \begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix} \begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix}$$

Therefore

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix}$$

But $(x_P, y_P, z_P) \in \mathbf{A}^3 \backslash \{(0,0,0)\}$. and hence x_P, y_P, z_P are not all 0. Hence our assumption that $x'_P = y'_P = z'_P = 0$ is wrong and x'_P, y'_P, z'_P are not all 0. Therefore the codomain of our restricted map can also be restricted. Hence our restricted map becomes $\tilde{\tau} : \mathbf{A}^3 \backslash \{(0,0,0\} \to \mathbf{A}^3 \backslash \{(0,0,0)\}$.

Composing the restricted map $\tilde{\tau}$ with the projection map π in Lemma 6.7 we obtain,

$$\mathbf{A}^3 \backslash \{(0,0,0\} \xrightarrow{\tilde{\tau}} \mathbf{A}^3 \backslash \{(0,0,0)\} \xrightarrow{\quad \pi \quad} \mathbf{P}^2$$

$$(x_p, y_p, z_p) \quad \mapsto \quad (x'_P, y'_P, z'_P) \quad \mapsto \quad [x'_P : y'_P : z'_P]$$

We obtain the map,

$$\begin{aligned} \pi \circ \tilde{\tau} : \mathbf{A}^3 \backslash \{(0,0,0)\} &\to \mathbf{P}^2 \\ (x_P, y_P, z_P) &\mapsto [x'_P : y'_P : z'_P] \end{aligned}$$

with

$$\begin{aligned} x'_P &= ax_P + by_P + cz_P \\ y'_P &= dx_P + ey_P + gz_P \\ z_P &= hx_P + jy_P + lz_P \end{aligned}$$

Let $(x_P, y_P, z_P) \sim (s_P, t_P, q_P)$ in $\mathbf{A}^3 \backslash \{(0,0,0)\}$.
Then $(s_P, t_P, q_P) = (\lambda x_P, \lambda y_P, \lambda z_P)$ for some non zero $\lambda \in K$.

$$\begin{aligned} (\pi \circ \tilde{\tau})(x_P, y_P, z_P) &= [x'_P : y'_P : z'_P] \\ &= [ax_P + by_P + cz_P : dx_P + ey_P + gz_P : hx_P + jy_P + lz_P] \end{aligned}$$

$$
\begin{aligned}
(\pi\circ\tilde{\tau})(s_P,t_P,q_P) &= (\pi\circ\tilde{\tau})(\lambda x_P,\lambda y_P,\lambda z_P)\\
&= [a\lambda x_P + b\lambda y_P + c\lambda z_P : d\lambda x_P + e\lambda y_P + g\lambda z_P : h\lambda x_P + j\lambda y_P + l\lambda z_P]\\
&= [\lambda(ax_P+by_P+cz_P) : \lambda(dx_P+ey_P+gz_P) : \lambda(hx_P+jy_P+lz_P)]\\
&= [ax_P+by_P+cz_P : dx_P+ey_P+gz_P : hx_P+jy_P+lz_P]
\end{aligned}
$$

Hence $(\pi\circ\tilde{\tau})(x_P,y_P,z_P) = (\pi\circ\tilde{\tau})(s_P,t_P,q_P)$. We can now construct the map $\tau : \mathbf{P}^2 \to \mathbf{P}^2$ by defining $\tau[x_P : y_P : z_P] = (\pi\circ\tilde{\tau})(x_P,y_P,z_P)$.

$$
\begin{array}{ccccc}
(x_P,y_P,z_P) & \in & \mathbf{A}^3\backslash\{(0,0,0)\} & \xrightarrow{\pi\circ\tilde{\tau}} & \mathbf{P}^2\\
\downarrow & & \downarrow & \overset{\tau}{\nearrow} & \\
[x_P : y_P : z_P] & \in & \mathbf{P}^2 & &
\end{array}
$$

Hence we see $\tilde{\tau}$ induces a well defined map $\tau : \mathbf{P}^2 \to \mathbf{P}^2$ which maps $[x_P : y_P : z_P] \mapsto [x'_P : y'_P : z'_P]$. We make note here that by definition $\tau[x_P : y_P : z_P] = (\pi\circ\tilde{\tau})(x_P,y_P,z_P)$. Hence $\tau(\pi(x_P,y_P,z_P)) = (\pi\circ\tilde{\tau})(x_P,y_P,z_P)$. Therefore $(\tau\circ\pi)(x_P,y_P,z_P)) = (\pi\circ\tilde{\tau})(x_P,y_P,z_P)$. Hence $\pi\circ\tilde{\tau} = \tau\circ\pi$.

(ii) $\mathbf{P}^2$ is covered by the open sets $U_0^2 \cup U_1^2 \cup U_2^2$. We can restrict the map τ to the open set U_0^2. Let $[x_P : y_P : z_P] \in U_0^2$. Then $\tau[x_P : y_P : z_P] = [ax_P+by_P+cz_P : dx_P+ey_P+gz_P : hx_P+jy_P+lz_P]$. The following diagram shows how we can construct a morphism from $U_0^2 \to \mathbf{P}^2$.

$$
\begin{array}{ccccccc}
[x_P : y_P : z_P] & \in & U_0^2 & & \mathbf{P}^2 & \ni & [r : s : t]\\
\downarrow & & \phi_0^2\downarrow & & \uparrow\pi & & \uparrow\\
(\frac{y_P}{x_P},\frac{z_P}{x_P}) & \in & \mathbf{A}^2 & \xrightarrow{\omega} & \mathbf{A}^3\backslash\{(0,0,0)\} & \ni & (r,s,t)\\
 & & \cup & & \cup & & \\
 & & (p,q) & \longmapsto & (a+bp+cq, & & \\
 & & & & d+ep+gq, & & \\
 & & & & h+jp+lq) & &
\end{array}
$$

By Lemma 5.44 $\phi_0^2 : U_0^2 \to \mathbf{A}^2$ is a morphism.
By Lemma 6.10 $\omega : \mathbf{A}^2 \to \mathbf{A}^3\backslash\{(0,0,0)\}$ is a morphism.
By Lemma 6.7 $\pi : \mathbf{A}^3\backslash\{(0,0,0)\} \to \mathbf{P}^2$ is a morphism.

$\pi \circ \omega \circ \phi_0^2$ is a morphism by Lemma 5.19.

$$\begin{aligned}
(\pi \circ \omega \circ \phi_0^2)[x_P : y_P : z_P] &= (\pi \circ \omega)(\phi_0^2[x_P : y_P : z_P]) \\
&= (\pi \circ \omega)(\frac{y_P}{x_P}, \frac{z_P}{x_P}) \\
&= \pi(\omega(\frac{y_P}{x_P}, \frac{z_P}{x_P})) \\
&= \pi(a + b\frac{y_P}{x_P} + c\frac{z_P}{x_P}, d + e\frac{y_P}{x_P} + g\frac{z_P}{x_P}, h + j\frac{y_P}{x_P} + l\frac{z_P}{x_P}) \\
&= [a + b\frac{y_P}{x_P} + c\frac{z_P}{x_P} : d + e\frac{y_P}{x_P} + g\frac{z_P}{x_P} : h + j\frac{y_P}{x_P} + l\frac{z_P}{x_P}] \\
&= [ax_P + by_P + cz_P : dx_P + ey_P + gz_P : hx_P + jy_P + lz_P]
\end{aligned}$$

Hence $\pi \circ \omega \circ \phi_0^2$ is equal to the restricted map $\tau : U_0^2 \to \mathbf{P}^2$. Therefore the restricted map $\tau : U_0^2 \to \mathbf{P}^2$ is a morphism.

We can restrict the map τ to the open set U_1^2. Let $[x_P : y_P : z_P] \in U_1^2$. Then $\tau[x_P : y_P : z_P] = [ax_P + by_P + cz_P : dx_P + ey_P + gz_P : hx_P + jy_P + lz_P]$. The following diagram shows how we can construct a morphism from $U_1^2 \to \mathbf{P}^2$.

$$\begin{array}{ccccccc}
[x_P : y_P : z_P] & \in & U_1^2 & & \mathbf{P}^2 & \ni & [r : s : t] \\
\downarrow & & \downarrow \phi_1^2 & & \uparrow \pi & & \uparrow \\
(\frac{x_P}{y_P}, \frac{z_P}{y_P}) & \in & \mathbf{A}^2 & \xrightarrow{\omega'} & \mathbf{A}^3 \backslash \{(0,0,0)\} & \ni & (r, s, t) \\
 & & \Cup & & \Cup & & \\
 & & (p, q) & \longmapsto & (ap + b + cq, & & \\
 & & & & dp + e + gq, & & \\
 & & & & hp + j + lq) & &
\end{array}$$

By Lemma 5.44 $\phi_1^2 : U_1^2 \to \mathbf{A}^2$ is a morphism.
By Lemma 6.10 $\omega' : \mathbf{A}^2 \to \mathbf{A}^3 \backslash \{(0,0,0)\}$ is a morphism.
By Lemma 6.7 $\pi : \mathbf{A}^3 \backslash \{(0,0,0)\} \to \mathbf{P}^2$ is a morphism.
$\pi \circ \omega' \circ \phi_1^2$ is a morphism by Lemma 5.19.

$$
\begin{aligned}
(\pi \circ \omega' \circ \phi_1^2)[x_P : y_P : z_P] = (\pi \circ \omega')(\phi_1^2[x_P : y_P : z_P])
&= (\pi \circ \omega')(\frac{x_P}{y_P}, \frac{z_P}{y_P}) \\
&= \pi(\omega'(\frac{x_P}{y_P}, \frac{z_P}{y_P})) \\
&= \pi(a\frac{x_P}{y_P} + b + c\frac{z_P}{y_P}, d\frac{x_P}{y_P} + e + g\frac{z_P}{y_P}, h\frac{x_P}{y_P} + j + l\frac{z_P}{y_P}) \\
&= [a\frac{x_P}{y_P} + b + c\frac{z_P}{y_P} : d\frac{x_P}{y_P} + e + g\frac{z_P}{y_P} : h\frac{x_P}{y_P} + j + l\frac{z_P}{y_P}] \\
&= [ax_P + by_P + cz_P : dx_P + ey_P + gz_P : hx_P + jy_P + lz_P]
\end{aligned}
$$

Hence $\pi \circ \omega' \circ \phi_1^2$ is equal to the restricted map $\tau : U_1^2 \to \mathbf{P}^2$. Therefore the restricted map $\tau : U_1^2 \to \mathbf{P}^2$ is a morphism.

We can restrict the map τ to the open set U_2^2. Let $[x_P : y_P : z_P] \in U_2^2$. Then $\tau[x_P : y_P : z_P] = [ax_P + by_P + cz_P : dx_P + ey_P + gz_P : hx_P + jy_P + lz_P]$. The following diagram shows how we can construct a morphism from $U_2^2 \to \mathbf{P}^2$.

$$
\begin{array}{ccccccc}
[x_P : y_P : z_P] & \in & U_2^2 & & \mathbf{P}^2 & \ni & [r : s : t] \\
\downarrow & & \downarrow \phi_2^2 & & \uparrow \pi & & \uparrow \\
(\frac{x_P}{z_P}, \frac{y_P}{z_P}) & \in & \mathbf{A}^2 & \xrightarrow{\omega''} & \mathbf{A}^3 \backslash \{(0,0,0)\} & \ni & (r,s,t) \\
& & \cup & & \cup & & \\
& & (p,q) & \longmapsto & (ap + bq + c, & & \\
& & & & dp + eq + g, & & \\
& & & & hp + jq + l) & &
\end{array}
$$

By Lemma 5.44 $\phi_2^2 : U_2^2 \to \mathbf{A}^2$ is a morphism.
By Lemma 6.10 $\omega'' : \mathbf{A}^2 \to \mathbf{A}^3 \backslash \{(0,0,0)\}$ is a morphism.
By Lemma 6.7 $\pi : \mathbf{A}^3 \backslash \{(0,0,0)\} \to \mathbf{P}^2$ is a morphism.
$\pi \circ \omega'' \circ \phi_2^2$ is a morphism by Lemma 5.19.

$$\begin{aligned}(\pi \circ \omega'' \circ \phi_2^2)[x_P : y_P : z_P] &= (\pi \circ \omega'')(\phi_2^2[x_P : y_P : z_P]) \\ &= (\pi \circ \omega'')(\frac{x_P}{z_P}, \frac{y_P}{z_P}) \\ &= \pi(\omega''(\frac{x_P}{z_P}, \frac{y_P}{z_P})) \\ &= \pi(a\frac{x_P}{z_P} + b\frac{y_P}{z_P} + c, d\frac{x_P}{z_P} + e\frac{y_P}{z_P} + g, h\frac{x_P}{z_P} + j\frac{y_P}{z_P} + l) \\ &= [a\frac{x_P}{z_P} + b\frac{y_P}{z_P} + c : d\frac{x_P}{z_P} + e\frac{y_P}{z_P} + g : h\frac{x_P}{z_P} + j\frac{y_P}{z_P} + l] \\ &= [ax_P + by_P + cz_P : \\ &\quad dx_P + ey_P + gz_P : \\ &\quad hx_P + jy_P + lz_P]\end{aligned}$$

Hence $\pi \circ \omega'' \circ \phi_2^2$ is equal to the restricted map $\tau : U_2^2 \to \mathbf{P}^2$. Therefore the restricted map $\tau : U_2^2 \to \mathbf{P}^2$ is a morphism.

By Lemma 6.4, $\tau : \mathbf{P}^2 \to \mathbf{P}^2$ is a morphism.

(iii) We change our map $\tilde{\tau}$ to matrix form for this part of the proof.

$$\tilde{\tau} : \mathbf{A}^3 \to \mathbf{A}^3$$
$$(x_P, y_P, z_P) \mapsto \begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix} \begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix}$$

The inverse map of $\tilde{\tau}$ is given as ,

$$\tilde{\tau}' : \mathbf{A}^3 \to \mathbf{A}^3$$
$$(x'_P, y'_P, z'_P) \mapsto \begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}^{-1} \begin{pmatrix} x'_P \\ y'_P \\ z'_P \end{pmatrix}$$

As the determinant of $\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}^{-1}$ is also non zero we can conclude from part (i) of the proof that $\tilde{\tau}'$ induces a well defined map

$$\tau' : \mathbf{P}^2 \to \mathbf{P}^2$$
$$[x'_P : y'_P : z'_P] \mapsto \pi(\begin{pmatrix} a & b & c \\ d & e & g \\ h & j & l \end{pmatrix}^{-1} \begin{pmatrix} x'_P \\ y'_P \\ z'_P \end{pmatrix})$$

where π is the projection map in Lemma 6.7. By part (ii) τ' is a morphism.
Let $[x_P : y_P : z_P] \in \mathbf{P}^2$.

$$\begin{aligned}(\tau' \circ \tau)[x_P : y_P : z_P] &= \tau'(\tau[x_P : y_P : z_P]) \\ &= \tau'(\tau(\pi(x_P, y_P, z_P))) \\ &= (\tau' \circ \tau \circ \pi)(x_P, y_P, z_P) \\ &= (\tau' \circ \pi \circ \tilde{\tau})(x_P, y_P, z_P) \\ &= (\pi \circ \tilde{\tau}' \circ \tilde{\tau})(x_P, y_P, z_P) \\ &= \pi(x_P, y_P, z_P) \\ &= [x_P : y_P : z_P]\end{aligned}$$

Hence $\tau' \circ \tau$ is the identity map on $\mathbf{P}^2$.
Let $[x'_P : y'_P : z'_P] \in \mathbf{P}^2$.

$$\begin{aligned}(\tau \circ \tau')[x'_P : y'_P : z'_P] &= (\tau \circ \tau')(\pi(x'_P, y'_P, z'_P)) \\ &= (\tau \circ \tau' \circ \pi)(x'_P, y'_P, z'_P) \\ &= (\tau \circ \pi \circ \tilde{\tau}')(x'_P, y'_P, z'_P) \\ &= (\pi \circ \tilde{\tau} \circ \tilde{\tau}')(x'_P, y'_P, z'_P) \\ &= \pi(x'_P, y'_P, z'_P) \\ &= [x'_P : y'_P : z'_P]\end{aligned}$$

Hence $\tau \circ \tau'$ is the identity map on $\mathbf{P}^2$.
Hence τ is an isomorphism.

(iv) Let $[x_P : y_P : z_P] \in Z_{\mathbf{P}^2}(f)$. Then

$$\begin{aligned}f(x_P, y_P, z_P) &= 0 \\ (f' \circ \tilde{\tau})(x_P, y_P, z_P) &= 0 \\ f'(\tilde{\tau}(x_P, y_P, z_P)) &= 0\end{aligned}$$

Hence $\tilde{\tau}(x_P, y_P, z_P) \in Z_{\mathbf{A}^3}(f')$. Therefore $\pi(\tilde{\tau}(x_P, y_P, z_P)) \in Z_{\mathbf{P}^2}(f')$.

$$\begin{aligned}\pi(\tilde{\tau}(x_P, y_P, z_P)) &= (\pi \circ \tilde{\tau})(x_P, y_P, z_P) \\ &= (\tau \circ \pi)(x_P, y_P, z_P) \\ &= \tau(\pi(x_P, y_P, z_P)) \\ &= \tau[x_P : y_P : z_P]\end{aligned}$$

Hence $\tau[x_P : y_P : z_P] \in Z_{\mathbf{P}^2}(f')$. Therefore $\tau(Z_{\mathbf{P}^2}(f)) \subseteq Z_{\mathbf{P}^2}(f')$.

(v) We are given f is irreducible. Assume f' is not irreducible. Then $f'(x',y',z') = g'(x',y',z')h'(x',y',z')$ for $g',h' \in K[x',y',z']$ both not constant. We are also given $f = f' \circ \tilde{\tau}$. Let $(x_P, y_P, z_P) \in \mathbf{A}^3$.

$$\begin{aligned} f(x_P, y_P, z_P) &= f'(\tilde{\tau}(x_P, y_P, z_P)) \\ &= f'(ax_P + by_P + cz_P, dx_P + ey_P + gz_P, hx_P + jy_P + lz_P) \end{aligned}$$

Hence

$$f(x,y,z) = f'(ax + by + cz, dx + ey + qz, hx + jy + lz)$$

Hence $f(x,y,z) = g'(ax+by+cz, dx+ey+qz, hx+jy+lz)h'(ax+by+cz, dx+ey+qz, hx+jy+lz)$ for $g'(ax+by+cz, dx+ey+qz, hx+jy+lz), h'(ax+by+cz, dx+ey+qz, hx+jy+lz) \in K[x,y,z]$.

Since f is irreducible, either $g'(ax+by+cz, dx+ey+qz, hx+jy+lz)$ or $h'(ax+by+cz, dx+ey+qz, hx+jy+lz)$ is constant.

Assume $g'(ax+by+cz, dx+ey+qz, hx+jy+lz) = t$ for $t \in K$ constant. Therefore $g'(\tilde{\tau}(x_P, y_P, z_P)) = t$ for all $(x_P, y_P, z_P) \in \mathbf{A}^3$. Let $(x'_P, y'_P, z'_P) \in \mathbf{A}^3$ be given. $\tilde{\tau}$ is a surjective map. Therefore $(x'_P, y'_P, z'_P) = \tilde{\tau}(x_Q, y_Q, z_Q)$ for some point $(x_Q, y_Q, z_Q) \in \mathbf{A}^3$. Hence $g'(x'_P, y'_P, z'_P) = g'(\tilde{\tau}(x_Q, y_Q, z_Q)) = t$. Hence $g'(x',y',z') = t$. Therefore $g'(x',y',z')$ is constant. But our initial assumption that f' is not irreducible means $g'(x',y',z')$ cannot be constant.
If we assume $h'(ax+by+cz, dx+ey+qz, hx+jy+lz) = s$ for some $s \in K$ constant, a similiar argument results in $h'(x',y',z')$ being constant which again contradicts our initial assumption that f' is not irreducible. This cannot happen. Therefore our initial assumption must be wrong meaning f' is irreducible.

By (iii) τ is an isomorphism. This means we have an inverse morphism τ' such that $\tau \circ \tau'$ is the identity on $\mathbf{P}^2_{(x',y',z')}$ and $\tau' \circ \tau$ is the Identity on $\mathbf{P}^2_{(x,y,z)}$. We are given $f = f' \circ \tilde{\tau}$ on $\mathbf{A}^3$. Hence $f \circ \tilde{\tau}' = f' \circ \tilde{\tau} \circ \tilde{\tau}'$. Therefore $f \circ \tilde{\tau}' = f'$.

By (i) and (ii) τ is a morphism with $\tau(Z(f)) \subseteq Z(f')$. Hence we can restrict the map τ to $\tau_{|Z(f)} : Z(f) \to Z(f')$. Since $Z(f)$ and $Z(f')$ are closed irreducible subsets of $\mathbf{P}^2$, $\tau_{|Z(f)}$ is a morphism by Lemma 5.24 and Lemma 6.5.
Let $[x_P : y_P : z_P] \in Z_{\mathbf{P}^2}(f')$. Then

$$\begin{aligned} f'(x'_P, y'_P, z'_P) &= 0 \\ (f \circ \tilde{\tau}')(x'_P, y'_P, z'_P) &= 0 \\ f(\tilde{\tau}'(x'_P, y'_P, z'_P)) &= 0 \end{aligned}$$

Hence $\tilde{\tau}'(x'_P, y'_P, z'_P) \in Z_{\mathbf{A}^3}(f)$. Therefore $\pi(\tilde{\tau}'(x'_P, y'_P, z'_P) \in Z_{\mathbf{P}^2}(f)$.

$$\begin{aligned}\pi(\tilde{\tau}'(x'_P, y'_P, z'_P)) &= (\pi \circ \tilde{\tau}')(x'_P, y'_P, z'_P) \\ &= (\tau' \circ \pi)(x'_P, y'_P, z'_P) \\ &= \tau'(\pi(x'_P, y'_P, z'_P)) \\ &= \tau'[x'_P : y'_P : z'_P]\end{aligned}$$

Hence $\tau'[x'_P : y'_P : z'_P] \in Z_{P^2}(f)$. Therefore $\tau'(Z_{P^2}(f')) \subseteq Z_{P^2}(f)$. We can restrict the map τ' to $\tau'_{|Z(f')} : Z(f') \to Z(f)$. $\tau'_{|Z(f')}$ is a morphism by Lemma 5.24 and Lemma 6.5.
Let $[x'_P : y'_P : z'_P] \in Z(f')$. Then

$$\begin{aligned}(\tau_{|Z(f)} \circ \tau'_{|Z(f')})[x'_P : y'_P : z'_P] &= \tau_{|Z(f)}(\tau'_{|Z(f')}[x'_P : y'_P : z'_P]) \\ &= \tau_{|Z(f)}(\tau'[x'_P : y'_P : z'_P]) \\ &= \tau(\tau'[x'_P : y'_P : z'_P]) \\ &= (\tau \circ \tau')[x'_P : y'_P : z'_P]) \\ &= [x'_P : y'_P : z'_P]\end{aligned}$$

Hence $\tau_{|Z(f)} \circ \tau'_{|Z(f')}$ is the identity map on $Z(f')$.
Let $[x_P : y_P : z_P] \in Z(f)$. Then

$$\begin{aligned}(\tau'_{|Z(f')} \circ \tau_{|Z(f)})[x_P : y_P : z_P] &= \tau'_{|Z(f')}(\tau_{|Z(f)}[x_P : y_P : z_P]) \\ &= \tau'_{|Z(f')}(\tau[x_P : y_P : z_P]) \\ &= \tau'(\tau[x_P : y_P : z_P]) \\ &= (\tau' \circ \tau)[x_P : y_P : z_P] \\ &= [x_P : y_P : z_P]\end{aligned}$$

Hence $\tau'_{|Z(f')} \circ \tau_{|Z(f)}$ is the identity map on $Z(f)$.
Hence $\tau_{|Z(f)} : Z(f) \to Z(f')$ is an isomorphism.

□

Theorem 6.14. *Any conic in $\mathbf{P}^2$ is isomorphic to $Z(y^2 - xz)$.*

Proof. By definition any conic on $\mathbf{P}^2$ is defined as the zero set of an irreducible homogenous polynomial of total degree 2. Let $f(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + gyz$ be an irreducible homogeneous polynomial of total degree 2.

$$\begin{aligned}f(x, y, z) &= ax^2 + by^2 + cz^2 + dxy + exz + gyz \\ &= ax^2 + exz + cz^2 + by^2 + dxy + gyz\end{aligned}$$

Let $h(r) \in K[r]$ be the polynomial ar^2+er+c. K is algebraically closed. Hence we can factor ar^2+er+c and obtain,

$$\begin{aligned} ar^2+er+c &= (tr+s)(pr+q) \\ &= tpr^2+trq+spr+sq \\ &= tpr^2+(tq+sp)r+sq \end{aligned}$$

Hence $a = tp, e = tq+sp, c = sq$. Substituting these equalities into $f(x, y, z)$ we get,

$$\begin{aligned} f(x, y, z) &= tpx^2+(tq+sp)xz+sqz^2+by^2+dxy+gyz \\ &= tpx^2+tqxz+spxz+sqz^2+by^2+dxy+gyz \\ &= tx(px+qz)+sz(px+qz)+by^2+dxy+gyz \\ &= (tx+sz)(px+qz)+by^2+dxy+gyz \end{aligned}$$

Case 1: $tx+sz$ and $px+qz$ are linearly dependent. By definition $i(tx+sz)+j(px+qz) = 0$ for some $i, j \in K$, with i and j not both 0. [SLLA, page 10].

Let $i = 0$. Then $j \neq 0$. Hence $0(tx+sz)+j(px+qz) = 0$. Therefore $j(px+qz) = 0$. Since $j \neq 0$, $px+qz$ is the zero vector. This means $f(x, y, z) = (tx+sz)(px+qz)+by^2+dxy+gyz = (tx+sz)(0)+by^2+dxy+gyz = by^2+dxy+gyz = y(by+dx+gz)$. Hence $f(x, y, z)$ would be reducible. Therefore $i \neq 0$.

Similarly, if we let $j = 0$. Then $tx+sz$ would be the zero vector. Hence $f(x, y, z) = (tx+sz)(px+qz)+by^2+dxy+gyz = 0(px+qz)+by^2+dxy+gyz = by^2+dxy+gyz = y(by+dx+gz)$. This again makes $f(x, y, z)$ reducible. Therefore $j \neq 0$.
Hence $i \neq 0 \neq j$. Therefore we have $px+qz = k(tx+sz)$ for some non zero $k \in K$, namely $k = -i/j$.

Therefore we have,

$$\begin{aligned} f(x, y, z) &= k(tx+sz)^2+by^2+dxy+gyz \\ &= k(tx+sz)^2+y(by+dx+gz) \\ &= ky'^2-kx'z' \\ &= k(y'^2-x'z') \end{aligned}$$

with $x' = -y/k, y' = tx+sz, z' = by+dx+gz$. So we have rewritten our polynomial $f \in K[x, y, z]$ as kf' with $f' = y'^2-x'z' \in K[x', y', z']$.

Assume x', y' and z' are linearly dependent in $K[x,y,z]$. Then by definition $k_1x' + k_2y' + k_3z' = 0$ for some $k_1, k_2, k_3 \in K$, with k_1, k_2, k_3 not all 0.
If $k_3 \neq 0$ then $k_3z' = -k_1x' - k_2y'$ in $K[x,y,z]$. Hence $z' = k_4x' + k_5y'$ in $K[x,y,z]$ with $k_4 = -k_1/k_3$ and $k_5 = -k_2/k_3$. Hence $f'(x', y'z') = y'^2 - x'(k_4x' + k_5y') = y'^2 - k_4x'^2 - k_5x'y'$. Put $g(r) = r^2 - k_5r - k_4 \in K[r]$. Since K is algebraically closed, $g(r)$ can be factored. Hence we obtain $r^2 - k_5r - k_4 = (r-a)(r+b) = r^2 - r(a-b) - ab$ for some $a, b \in K$. Therefore $a - b = k_5$ and $ab = k_4$. Substituting these equations into $f'(x', y', z')$ we get $f'(x', y', z') = y'^2 - (a-b)x'y' - abx'^2 = y'^2 - ax'y' + bx'y' - abx'^2 = y'(y' - ax') + bx'(y' - ax') = (y' - ax')(y' + bx')$ in $K[x,y,z]$. Therefore $kf'(x', y', z') = k(y' - ax')(y' + bx')$. Hence $f(x,y,x) = k(tx + sz - a(-y/k))(tx + sz + b(-y/k))$. Hence f is not irreducible which is a contradiction. Hence x', y' and z' must be linearly independent in $K[x,y,z]$. We obtain the following map.

$$\tilde{\varphi} : \mathbf{A}^3 \to \mathbf{A}^3$$
$$(x_P, y_P, z_P) \mapsto (\frac{-y_P}{k}, tx_P + sz_P, by_P + dx_P + gz_P)$$

with $\det \begin{pmatrix} 0 & \frac{-1}{k} & 0 \\ t & 0 & s \\ d & b & g \end{pmatrix} = 0 \begin{pmatrix} 0 & s \\ b & g \end{pmatrix} + \frac{1}{k} \begin{pmatrix} t & s \\ d & g \end{pmatrix} + 0 \begin{pmatrix} t & 0 \\ d & b \end{pmatrix} = 0 + \frac{1}{k}(tg - sd) + 0 = \frac{1}{k}(tg - sd)$, $\frac{1}{k} \neq 0$ $tg - sd \neq 0$ since $tx + sz, by + dx + gz$ are linearly independent. By construction, $f = kf' \circ \tilde{\varphi}$. f, kf' both homogeneous. Hence by Theorem 6.13 part (v), f' is irreducible and $Z_{\mathbf{P}^2}(f) \cong Z_{\mathbf{P}^2}(kf')$. Note $Z_{\mathbf{P}^2}(kf') = Z_{\mathbf{P}^2}(f')$. Hence $Z_{\mathbf{P}^2}(f) \cong Z_{\mathbf{P}^2}(f')$.

Case 2. When $tx + sz$ and $px + qz$ are linearly independant. In matrix form,

$$f(x,y,z) = (tx + sz)(px + qz) + by^2 + \begin{pmatrix} d & g \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} y$$

Put $x' = tx + sz$, $z' = px + qz$ and $y' = y$. Hence

$$\begin{pmatrix} x' \\ z' \end{pmatrix} = \begin{pmatrix} t & s \\ p & q \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

Therefore

$$\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} t & s \\ p & q \end{pmatrix}^{-1} \begin{pmatrix} x' \\ z' \end{pmatrix}$$

Hence $f(x,y,z) \in K[x,y,z]$ can now be expressed as a polynomial in $K[x', y', z']$

as follows,

$$\begin{aligned} f'(x',y',z') &= x'z' + by'^2 + \begin{pmatrix} d & g \end{pmatrix} \begin{pmatrix} t & s \\ p & q \end{pmatrix}^{-1} \begin{pmatrix} x' \\ z' \end{pmatrix} y' \\ &= x'z' + \alpha x'y' + \beta z'y' + by'^2 \end{aligned}$$

where α, β are given by

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} d & g \end{pmatrix} \begin{pmatrix} t & s \\ p & q \end{pmatrix}^{-1}$$

We obtain the following map.

$$\begin{aligned} \tilde{\varphi'} &: \mathbf{A}^3 \to \mathbf{A}^3 \\ (x_P, y_P, z_P) &\mapsto (tx_P + sz_P, y_P, px_P + qz_P) \end{aligned}$$

with $\det \begin{pmatrix} t & 0 & s \\ 0 & 1 & 0 \\ p & 0 & q \end{pmatrix} = 0 \begin{pmatrix} 0 & s \\ 0 & q \end{pmatrix} - 1 \begin{pmatrix} t & s \\ p & q \end{pmatrix} + 0 \begin{pmatrix} t & 0 \\ p & 0 \end{pmatrix} = 0 - 1(tq - sp) + 0 = sp - tq \neq 0$ because $tx + sz, px + qz$ are linearly independant. By construction, $f = f' \circ \tilde{\varphi'}$. f, f' both homogeneous. By Theorem 6.13 part (v) if f is irreducible then f' is irreducible and $Z_{\mathbf{P}^2}(f) \cong Z_{\mathbf{P}^2}(f')$. Now we have

$$\begin{aligned} f'(x',y',z') &= x'z' + \alpha x'y' + \beta z'y' + by'^2 \\ &= (x' + \beta y')(z' + \alpha y') - \alpha\beta y'^2 + by'^2 \\ &= (x' + \beta y')(z' + \alpha y') - y'^2(\alpha\beta - b) \end{aligned}$$

$\alpha\beta - b \neq 0$. Otherwise f' is reducible. But f' is irreducible by Theorem6.13 part (v). Put $\alpha\beta - b = \gamma$. Then $f'(x',y',z') = (x' + \beta y')(z' + \alpha y') - y'^2\gamma$. Put $\overline{f'}(x',y',z') = -(\frac{x'+\beta y'}{\gamma})(z' + \alpha y') + y'^2$. Note that $Z_{\mathbf{P}^2}(f') = Z_{\mathbf{P}^2}(\overline{f}')$. Hence $Z_{\mathbf{P}^2}(f) \cong Z_{\mathbf{P}^2}(\overline{f'})$. Let $x'' = \frac{x'+\beta y'}{\gamma}$, $z'' = z' + \alpha y'$ and $y'' = y'$. Hence $\overline{f'} \in K[x',y',z']$ can now be written as a polynomial in $K[x'',y'',z'']$ as follows, $f''(x'',y''.z'') = y''^2 - x''z''$. We now obtain the following map.

$$\begin{aligned} \tilde{\varphi''} &: \mathbf{A}^3 \to \mathbf{A}^3 \\ (x'_P, y'_P, z'_P) &\mapsto (\frac{x'_P + \beta y'_P}{\gamma}, y'_P, z'_P + \alpha y'_P) \end{aligned}$$

with $\det \begin{pmatrix} \frac{1}{\gamma} & \frac{\beta}{\gamma} & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{pmatrix} = 0 \begin{pmatrix} \frac{\beta}{\gamma} & 0 \\ \alpha & 1 \end{pmatrix} \text{-}1 \begin{pmatrix} \frac{1}{\gamma} & 0 \\ 0 & 1 \end{pmatrix} = 0 \begin{pmatrix} \frac{1}{\gamma} & \frac{\beta}{\gamma} \\ 0 & \alpha \end{pmatrix} = 0 - 1(\frac{1}{\gamma} \cdot 1 - 0 \cdot 0) - 0 = \frac{-1}{\gamma} \neq 0$. By construction, $\overline{f'} = f'' \circ \tilde{\varphi''}$. $\overline{f'}, f''$ both homogeneous. By

Theorem 6.13 part (v) $Z_{\mathbf{P}^2}(\overline{f'}) \cong Z_{\mathbf{P}^2}(f'')$. Hence we have $Z_{\mathbf{P}^2}(f) \cong Z_{\mathbf{P}^2}(f') = Z_{\mathbf{P}^2}(\overline{f'}) \cong Z_{\mathbf{P}^2}(f'')$. Therefore $Z_{\mathbf{P}^2}(f) \cong Z_{\mathbf{P}^2}(f'')$.

Hence we have that any conic on $\mathbf{P}^2$ is isomorphic to $Z(y^2 - xz)$. □

Theorem 6.15. *Any conic in* $\mathbf{P}^2$ *is isomorphic to* $\mathbf{P}^1$.

Proof. By Theorem 6.14 any conic on $\mathbf{P}^2$ is isomorphic to $Z(y^2 - xz)$. By Theorem 6.6 $Z(y^2 - xz) \cong \mathbf{P}^1$. Hence any conic on $\mathbf{P}^2$ is isomorphic to $\mathbf{P}^1$. □

We now show an example. Before this we define the following.

Definition 6.16. The *characteristic* of a ring R, often denoted $\text{char}(R)$, is defined to be the smallest positive number of copies of the ring's multiplicative identity 1 that will sum to the additive identity 0. If no such number exists, the ring is said to have characteristic zero.

Example 6.17. Let $f(x, y, z) = x^2 + y^2 + z^2$. Assume characteristic of K is not 2. $Z(x^2 + y^2 + z^2) \cong \mathbf{P}^1$.

Proof. First we will show $f(x, y, z)$ is an irreducible homogeneous polynomial of total degree 2. It is clear $f(x, y, z)$ is a homogeneous polynomial of total degree 2. Assume $f(x, y, z)$ is not irreducible. Then $x^2 + y^2 + z^2 = (ax + by + cz)(dx + ey + gz)$ for $a, b, c, d, e, g \in K, a \neq 0$. Without loss of generality we can assume $a = 1$. Hence

$$x^2 + y^2 + z^2 = (x + by + cz)(dx + ey + gz)$$
$$x^2 + y^2 + z^2 = dx^2 + (e + bd)xy + (g + cd)xz + (bg + ce)yz + bey^2 + cgz^2$$

Therefore we obtain the following equations. $d = 1, e + bd = 0, g + cd = 0, bg + ce = 0, be = 1$ and $cg = 1$. Replacing $d = 1$ where it applies we obtain $e + b = 0, g + c = 0$. Hence we look at the following system of equations.

$$\begin{aligned} e + b &= 0 \\ g + c &= 0 \\ bg + ce &= 0 \\ be &= 1 \\ cg &= 1 \end{aligned}$$

Since K is a field, $be = 1$ means $b \neq 0$ and $e \neq 0$ and $cg = 1$ means $c \neq 0$ and $g \neq 0$. From $e + b = 0$ we get $e = -b$. From $g + c = 0$ we get $g = -c$. Substituting into $bg + ce = 0$ we obtain $b(-c) + c(-b) = 0$. This results in $-2bc = 0$. By assumption the characteristic of K is not 2. Hence either $b = 0$ or $c = 0$. We have already shown that $b \neq 0$ and $c \neq 0$. Hence $x^2 + y^2 + z^2$ is irreducible.
$x^2 + y^2 + z^2 = x^2 + 0xz + z^2 + 0xy + 0yz + y^2$. Let $h(r) \in K[r]$ be the polynomial

ar^2+er+c. In this case $ar^2+er+c = r^2+0r+1$. r^2+0r+1 can be factored as follows,

$$r^2+0r+1 = (r-i)(r+i)$$

where $i^2=-1$. Therefore $f(x,y,z) = (x-iz)(x+iz)+y^2$. $x-iz$ and $x+iz$ are linearly independant. To show this we simply check the determinant of the matrix, $\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}$ Determinant is given by $(1\cdot i)-(1\cdot -i) = i+i = 2i \neq 0$. Hence we are in case 2 of Theorem 6.14. In matrix form,

$$f(x,y,z) = (x-iz)(x+iz)+y^2+\begin{pmatrix}0 & 0\end{pmatrix}\begin{pmatrix}x\\ z\end{pmatrix}y$$

Put $x' = x-iz$, $z' = x+iz$ and $y' = y$. Hence

$$\begin{pmatrix}x'\\ z'\end{pmatrix} = \begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\begin{pmatrix}x\\ z\end{pmatrix}$$

Therefore

$$\begin{pmatrix}x\\ z\end{pmatrix} = \begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}^{-1}\begin{pmatrix}x'\\ z'\end{pmatrix}$$

Hence $f(x,y,z)\in K[x,y,z]$ can now be expressed as a polynomial in $K[x',y',z']$ as follows,

$$\begin{aligned}f'(x',y',z') &= x'z'+y'^2+\begin{pmatrix}0 & 0\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}^{-1}\begin{pmatrix}x'\\ z'\end{pmatrix}y'\\ &= x'z'+\alpha x'y'+\beta z'y'+y'^2\end{aligned}$$

where α,β are given by

$$\begin{pmatrix}\alpha & \beta\end{pmatrix} = \begin{pmatrix}0 & 0\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}^{-1}$$

In other words $\alpha = 0, \beta = 0$. We obtain the following map.

$$\begin{aligned}\tilde{\varphi}' &: \mathbf{A}^3 \to \mathbf{A}^3\\ (x_P, y_P, z_P) &\mapsto (x_P-iz_P, y_P, x_P+iz_P)\end{aligned}$$

with $\det\begin{pmatrix}1 & 0 & -i\\ 0 & 1 & 0\\ 1 & 0 & i\end{pmatrix} = 0\begin{pmatrix}0 & -i\\ 0 & i\end{pmatrix} - 1\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix} + 0\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = -1(2i) = -2i \neq 0$. By construction, $f = f'\circ\tilde{\varphi}'$. f, f' both homogeneous. f is irreducible hence

f' is irreducible. Hence by Theorem 6.13 part (v) $Z_{\mathbf{P}^2}(f) \cong Z_{\mathbf{P}^2}(f')$. Now we have

$$\begin{aligned} f'(x',y',z') &= x'z' + 0x'y' + 0z'y' + y'^2 \\ &= (x'+0y')(z'+0y') - 0 \cdot 0y'^2 + y'^2 \\ &= (x'+0y')(z'+0y') - y'^2(0 \cdot 0 - 1) \\ &= (x'+0y')(z'+0y') - y'^2(-1) \end{aligned}$$

Put $\gamma = -1$. Now $f'(x',y',z') = (x'+0y')(z'+0y') - y'^2(\gamma)$. Put $\overline{f'}(x',y',z') = -(\frac{x'+0y'}{\gamma})(z'+0y') + y'^2$. Note that $Z_{P^2}(f') = Z_{P^2}(\overline{f'})$. Hence $Z_{\mathbf{P}^2}(f) \cong Z_{\mathbf{P}^2}(\overline{f'})$. Let $x'' = \frac{x'+0y'}{\gamma}$, $z'' = z'+0y'$ and $y'' = y'$. Hence $\overline{f'}$ can be expressed as $f''(x'',y''.z'') = y''^2 - x''z''$. We obtain the following map.

$$\begin{aligned} \tilde{\varphi''} &: \mathbf{A}^3 \to \mathbf{A}^3 \\ (x'_P, y'_P, z'_P) &\mapsto (\frac{x'_P + 0y'_P}{-1}, y'_P, z'_P + 0y'_P) \end{aligned}$$

with det $\begin{pmatrix} \frac{1}{-1} & \frac{0}{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - 1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + 0\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = -1(-1) = 1 \neq 0$. By construction, $\overline{f'} = f'' \circ \tilde{\varphi''}$. $\overline{f'}, f''$ both homogeneous. $\overline{f'}$ is irreducible hence f'' is irreducible. By Theorem 6.13 part (v) $Z_{\mathbf{P}^2}(\overline{f'}) \cong Z_{\mathbf{P}^2}(f'')$. Hence we have $Z_{\mathbf{P}^2}(f) \cong Z_{\mathbf{P}^2}(f') = Z_{P^2}(\overline{f'}) \cong Z_{\mathbf{P}^2}(f'')$. Therefore $Z_{\mathbf{P}^2}(f) \cong Z_{\mathbf{P}^2}(f'')$. Hence we have $Z(x^2+y^2+z^2) \cong Z(xz-y^2)$. By Theorem 6.6 $Z(xz-y^2) \cong \mathbf{P}^1$. Hence $Z(x^2+y^2+z^2) \cong \mathbf{P}^1$. □

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