EINSTEIN–HERMITIAN CONNECTION ON TWISTED HIGGS BUNDLES

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ABSTRACT. Let $X$ be a smooth projective variety over $\mathbb{C}$. We prove that a twisted Higgs vector bundle $(\mathcal{E}, \theta)$ on $X$ admits an Einstein–Hermitian connection if and only if $(\mathcal{E}, \theta)$ is polystable. A similar result for twisted vector bundles (no Higgs fields) was proved in [10]. Our approach is simpler.

RÉSUMÉ. Connexions d’Einstein–Hermite sur les fibrés de Higgs tordus. Soit $X$ une variété projective lisse sur $\mathbb{C}$. Nous démontrons qu’un fibré de Higgs tordu $(\mathcal{E}, \theta)$ sur $X$ possède une connexion d’Einstein–Hermite si et seulement si $(\mathcal{E}, \theta)$ est polystable. Un résultat analogue pour les fibrés vectoriels (dépourvus d’un champ de Higgs) a été démontré dans [10]. Notre approche est plus simple.

1. Introduction

Donaldson and Uhlenbeck–Yau proved that a vector bundle on a complex projective manifold admits an Einstein–Hermitian connection if and only if it is polystable [3], [9]. A generalization of Einstein–Hermitian connections for Higgs bundles was formulated by Hitchin (for curves) and Simpson (higher dimensions). They proved that a Higgs bundle $(\mathcal{E}, \theta)$ admits an Einstein–Hermitian connection if and only if it is polystable [4], [8].

Our aim here is to establish a similar result for twisted sheaves on a smooth complex projective variety. Let $X$ be an irreducible smooth projective variety over $\mathbb{C}$. A twisted vector bundle on $X$ is a pair $(\mathcal{X}, \mathcal{E})$, where

$$\mathcal{X} \rightarrow X$$

is a gerbe banded by $\mu_n$ (the $n$–th roots of unity) for some $n$, and $\mathcal{E}$ is a vector bundle over $\mathcal{X}$; see [7], [6], [5], [11] for twisted bundles. A twisted Higgs bundle on $X$ is a twisted vector bundle together with a Higgs field on it.

We prove that a twisted Higgs bundle on $X$ admits an Einstein–Hermitian connection if and only if it is polystable (see Theorem 3.1).

Let $G$ be a connected reductive linear algebraic group defined over $\mathbb{C}$. Theorem 3.1 generalizes to twisted Higgs principal $G$–bundles (this is explained at the end).

In [10], Wang proved a similar result for twisted vector bundles without Higgs structure.

2. Twisted Higgs bundles

The base field will be $\mathbb{C}$. For any positive integer $n$, by $\mu_n$ we will denote the finite subgroup of $\mathbb{C}^*$ consisting of the $n$–th roots of 1.

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Let $X$ be an irreducible smooth projective variety. Let

\[(2.1) \quad f : \mathcal{X} \rightarrow X\]

be a gerbe banded by $\mu_n$. The cotangent bundle of $\mathcal{X}$ will be denoted by $\Omega^1_{\mathcal{X}}$. For any nonnegative integer $i$, let $\Omega^i_{\mathcal{X}} := \bigwedge^i \Omega^1_{\mathcal{X}}$ be the $i$-th exterior power.

Let

\[\mathcal{E} \rightarrow \mathcal{X}\]

be a vector bundle. Let $\text{End}(\mathcal{E}) := \mathcal{E} \otimes \mathcal{E}^*$ be the endomorphism bundle. The associative algebra structure of $\text{End}(\mathcal{E})$ and the exterior algebra structure of $\bigoplus_{i \geq 0} \Omega^i_{\mathcal{X}}$ together define an algebra structure on $\text{End}(\mathcal{E}) \otimes \bigoplus_{i \geq 0} \Omega^i_{\mathcal{X}}$.

A Higgs field on $\mathcal{E}$ is a section $\theta$ of $\text{End}(\mathcal{E}) \otimes \Omega^1_{\mathcal{X}}$ such that the section $\theta \wedge \theta$ of $\text{End}(\mathcal{E}) \otimes \Omega^2_{\mathcal{X}}$ vanishes identically.

A Higgs bundle on $X$ is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a vector bundle on $X$, and $\theta$ is a Higgs field on $\mathcal{E}$. A Higgs bundle on $X$ will be called a twisted Higgs bundle on $X$. Given a Higgs bundle $(\mathcal{E}, \theta)$ on $X$, a coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ will be called a Higgs subsheaf if $\theta(\mathcal{F}) \subset \mathcal{F} \otimes \Omega^1_{\mathcal{X}}$.

Let $G$ be a complex linear algebraic group. A Higgs $G$–bundle on $X$ is a principal $G$–bundle $E_G \rightarrow X$ and a section $\beta \in H^0(X, \text{ad}(E_G) \otimes \Omega^1_{\mathcal{X}})$ such that $\beta \wedge \beta = 0$, where $\text{ad}(E_G)$ is the adjoint vector bundle.

Fix a very ample line bundle $L$ over $X$. The degree of a torsionfree coherent sheaf $\mathcal{F}$ on $X$ will be defined to be $\text{degree}((\det \mathcal{F})^\otimes n)/n^2 \in \mathbb{Q}$. Note that $(\det \mathcal{F})^\otimes n$ descends to a line bundle on $X$; its degree is computed using $L$. Fix a Kähler form $\omega_X$ on $X$ representing $c_1(L)$. Since the morphism $f$ in (2.1) is étale, the pullback

\[(2.2) \quad \omega_X := f^*\omega_X\]

is a Kähler form on $\mathcal{X}$. A Higgs bundle $(\mathcal{E}, \theta)$ is called stable (respectively, semistable) if for every Higgs subsheaf $\mathcal{F}$ with $1 \leq \text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})$, the inequality

\[
\frac{\text{degree}(\mathcal{F})}{\text{rank}(\mathcal{F})} < \frac{\text{degree}(\mathcal{E})}{\text{rank}(\mathcal{E})}\quad (\text{respectively,} \quad \frac{\text{degree}(\mathcal{F})}{\text{rank}(\mathcal{F})} \leq \frac{\text{degree}(\mathcal{E})}{\text{rank}(\mathcal{E})})
\]

holds. A semistable Higgs bundle is called polystable if it is a direct sum of stable Higgs bundles.

For any vector bundle $\mathcal{E} \rightarrow \mathcal{X}$, we have a decomposition $\mathcal{E} = \bigoplus_{\chi \in \mu^*_n} \mathcal{E}_\chi$. Henceforth, we will consider vector bundles $\mathcal{E}$ with $\mathcal{E}_\chi \neq 0$ for at most one character $\chi$.

Define the homomorphism

\[(2.3) \quad \rho : \text{GL}(r, \mathbb{C}) \rightarrow \text{PGL}(r, \mathbb{C}) \times \mathbb{G}_m =: H\]

by sending $A$ to the class of $A$ and to $(\det A)^n$. Given a vector bundle $\mathcal{E} \rightarrow \mathcal{X}$, the extension of its structure group along $\rho$ defines a principal $H$–bundle $\mathcal{E}_H \rightarrow \mathcal{X}$. Since the inertia $\mu_n$ acts trivially on $\mathcal{E}_H$, it descends to a principal $H$–bundle $E_H \rightarrow X$. A Higgs field $\theta$ on $\mathcal{E}$ induces a Higgs field $\theta_H$ on $\mathcal{E}_H$. This Higgs field $\theta_H$ on $\mathcal{E}_H$ descends to a Higgs field on $E_H$, which we again denote by $\theta_H$. 
The definitions of Higgs (semi)stable and polystable principal bundles are recalled in [1, p. 551], [2].

**Lemma 2.1.** A Higgs bundle \((E, \theta)\) on \(X\) is polystable if and only if the induced Higgs \(H\)-bundle \((E_H, \theta_H)\) on \(X\) is polystable.

**Proof.** The central isogeny \(\rho\) in (2.3) produces a bijection of parabolic subgroups. For any parabolic subgroup \(P \subset \text{GL}(r, \mathbb{C})\), there is a natural bijective correspondence between the reductions of structure group of the principal \(\text{GL}(r, \mathbb{C})\)-bundle \(E\) to \(P\) over any open subset \(f^{-1}(U)\) and the reductions of structure group of the principal \(H\)-bundle \(E_H\) to \(\rho(P)\) over \(U\). This bijection proves the lemma. \(\Box\)

### 3. Einstein–Hermitian connection on polystable twisted Higgs bundles

A *Hermitian structure* on a vector bundle \(E\) on \(X\) is a smooth inner product on the fibers which is invariant under the action of \(\mu_n\) on the fibers of \(E\). A Hermitian structure on \(E\) produces a \(C^\infty\) complex connection on \(E\). Let \((E, \theta)\) be a Higgs bundle. An *Einstein–Hermitian connection* on \((E, \theta)\) is a Hermitian structure on \(E\) such that corresponding connection \(\nabla\) on \(E\) has the following property:

\[
\Lambda_{\omega_X}(\text{Curv}(\nabla) + [\theta, \theta^*]) = c \cdot \text{Id}_E,
\]

for some constant scalar \(c\), where \(\Lambda_{\omega_X}\) is the adjoint of multiplication by the Kähler form \(\omega_X\) (see (2.2)), \(\text{Curv}(\nabla)\) is the curvature of \(\nabla\), and \(\theta^*\) is the adjoint of \(\theta\) constructed using the Hermitian form on \(E\).

**Theorem 3.1.** Let \((E, \theta)\) be a twisted Higgs bundle on \(X\). Then \((E, \theta)\) is polystable if and only if it admits an Einstein–Hermitian connection.

**Proof.** Let \((E, \theta)\) be a Higgs bundle on \(X\). First assume that \((E, \theta)\) is polystable. From Lemma 2.1 we know that the induced Higgs \(H\)-bundle \((E_H, \theta_H)\) on \(X\) is polystable. A polystable Higgs \(H\)-bundle on \(X\) admits an Einstein–Hermitian connection [8], [1]. Since \((E_H, \theta_H)\) is the descent of \((E_H, \theta_H)\), an Einstein–Hermitian connection on \((E_H, \theta_H)\) produces an Einstein–Hermitian connection on \((E_H, \theta_H)\). A connection on \(E_H\) defines connection on \(E\) because the homomorphism of Lie algebras

\[
\text{Lie}(\text{GL}(r, \mathbb{C})) \rightarrow \text{Lie}(H)
\]

induced by the homomorphism \(\rho\) in (2.3) is an isomorphism. The connection on \((E, \theta)\) induced by an Einstein–Hermitian connection on \((E_H, \theta_H)\) is clearly Einstein–Hermitian.

Conversely, an Einstein–Hermitian connection on \((E, \theta)\) induces an Einstein–Hermitian connection on the associated Higgs \(H\)-bundle \((E_H, \theta_H)\), which, in turn, induces an Einstein–Hermitian connection on the descended Higgs \(H\)-bundle \((E_H, \theta_H)\). Therefore, the Higgs \(H\)-bundle \((E_H, \theta_H)\) is polystable. Hence from Lemma 2.1 we conclude that the Higgs bundle \((E, \theta)\) is polystable. \(\Box\)

Let \(G\) be a connected reductive linear algebraic group defined over \(\mathbb{C}\). Let \(Z\) be the center of \(G\); define \(G' := [G', G]\).
The above theorem holds for principal Higgs $G$–bundles on $\mathcal{X}$. The proof is the same, but, instead of the homomorphism (2.3), we use the homomorphism

$$\rho : G \longrightarrow H := G/Z \times (G/G') : g \mapsto (p(g), q(g)^n),$$

where $p : G \longrightarrow G/Z$ and $q : G \longrightarrow G/G'$ are the natural projections. Note that $G/G' \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*$.

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