ON SEMISTABLE VECTOR BUNDLES OVER CURVES

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Abstract. Let X be a geometrically irreducible smooth projective curve defined over a field k, and let E be a vector bundle on X. Then E is semistable if and only if there is a vector bundle F on X such that $H^i(X, F \otimes E) = 0$ for $i = 0, 1$. We give an explicit bound for the rank of F. The proof uses a result of Popa for the case that k is algebraically closed.

Résumé. **Sur les fibrés vectoriels semi-stable au-dessus des courbes.** Soit X une courbe projective lisse géométriquement irréductible définie au-dessus d'un corps k, et soit E un fibré vectoriel sur X. E est semi-stable si et seulement s'il y a un fibré vectoriel F sur X tel que $H^i(X, F \otimes E) = 0$ pour $i = 0, 1$. Nous donnons une borne explicite pour le rang de F. La preuve utilise un résultat de Popa pour le cas oú k est algébriquement clos.

1. Introduction

Let k be field with algebraic closure K, and let X be a geometrically irreducible, smooth, projective curve defined over k of genus $g \geq 2$. Recall that a vector bundle E over X is called semistable if for all subbundles of positive rank $E' \subset E$ defined over k, the inequality $\mu(E') \leq \mu(E)$ holds. Here the rational number $\mu(E') := \frac{\deg(E')}{\mathrm{rk}(E')}$ is the *slope* of the vector bundle E'. It is known that E is semistable if and only if the base change $E \otimes_k K \longrightarrow X \times_k K$ is semistable; this is proved in [4, p. 97, Proposition 3].

Assume that there exists a second vector bundle F on X, such that $H^*(X, F \otimes E) = 0$, meaning $H^0(X, F \otimes E) = 0 = H^1(X, F \otimes E)$. Such a vector bundle F we call *cohomologically orthogonal* to E. This implies that $\chi(F \otimes E) = 0$, or equivalently, $\mu(F) + \mu(E) = g - 1$. If there were a destabilizing bundle $E' \subset E$, then we would have $\mu(F \otimes E') > g - 1$ implying $h^0(F \otimes E') > 0$. This is absurd because $H^0(F \otimes E') \subset H^0(F \otimes E) = 0$. Consequently, the statement $H^*(X, F \otimes E) = 0$ implies the semistability of E (and of F as well).

Faltings showed in [2] that for k algebraically closed, the converse is also true: if E is semistable, then there exists a vector bundle F with $H^*(X, F \otimes E) = 0$. Popa showed in [5, Theorem 5.3], that F can be chosen to have a prescribed rank and determinant that depend only on the rank and degree of E.

Faltings' result generalizes to arbitrary fields k as follows. Given the semistable vector bundle E on X defined over k, it yields a cohomologically orthogonal bundle F' defined over K. This F' is then defined over some finite extension ℓ/k. The pushforward F of F' along the morphisms $X \times_k \ell \longrightarrow X$ gives us a vector bundle F defined over k, which is cohomologically orthogonal to E according to the projection formula.

For perfect fields k, a bound on the rank of F is given in [1]. The main result, namely Theorem 3.1, of [1] shows that for a perfect field k and a semistable vector bundle E on

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X/k there exists a vector bundle F of a given rank R defined over k such that $H^*(X, F \otimes E) = 0$. However, the rank R of F is huge in general, and the bound in [1] is far from being optimal.

The purpose of this paper is twofold: First we remove the perfectness assumption. Secondly, we improve the bound on the rank R of the sheaf F which is cohomologically orthogonal to a semistable E.

In [1], a point outside the divisor Θ_E was constructed using [1, corollary 2.5] and the fact that the moduli space of S-equivalence classes of rank R vector bundles on X is projective. Here we use the geometry of the moduli space of rank R bundles with fixed determinant, which is known to be a unirational variety.

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2. Notation

k	–	a field
K	–	its algebraic closure
X	–	a smooth, projective curve defined over the field k which is geometrically irreducible
g	–	the genus of X
ω_X	–	the dualizing line bundle on X
E	–	a vector bundle on X defined over k
r	–	the rank $\mathrm{rk}(E)$ of E
d	–	the degree $\deg(E)$ of E
h	–	$h := \gcd(r, d)$, the greatest common divisor of r and d
m	–	$m := \lceil \frac{r^2+1}{8h} \rceil$ the round up
R	–	$R := 2rm$ (this will be the rank of a cohomologically orthogonal bundle F over k or some finite extension L/k if k is a finite field)
D	–	$D := m(2r(g-1) - 2d)$ (this will be the degree of F)
L	–	$L := \omega_X^{\otimes mr} \otimes \det(E)^{\otimes -2m}$ (this will be the determinant of the bundle F).

3. Infinite fields

Theorem 1. *Suppose k is a infinite field, and let X be a smooth projective geometrically irreducible curve over k. For a vector bundle E of rank r and degree d over X, the following three statements are equivalent:*

(i) *The vector bundle E is semistable.*
(ii) *There exists a vector bundle F on X defined over k such that $H^*(X, F \otimes E) = 0$.*
(iii) *There exists a vector bundle F on X defined over k of rank R and determinant L such that $H^*(X, F \otimes E) = 0$.*

Proof. Note that (iii) $\Longrightarrow$ (ii) is trivial, and (ii) $\Longleftrightarrow$ (i) was discussed in the introduction. We will show that (i) $\Longrightarrow$ (iii). Here are the steps of the proof.

(1) Since the statement is twist invariant we may assume (replacing E by $E \otimes \omega_X^{\otimes n}$ for an appropriate integer n) that $2 - 3g \le \mu(E) < -g$.

(2) We take R and L as above, and let F be any semistable vector bundle of rank R and determinant L defined over K. We obtain $\mu(F) = g-1-\mu(E) > 2g-1$. Thus, for any point $P \in X(K)$ we have $\mu(F(-P)) > 2g-2$. By semistability we conclude that $\mathrm{Hom}(F(-P), \omega_X) = 0$. From Serre duality we have $H^1(X, F(-P)) = 0$. Therefore, it follows that any semistable bundle F of rank R and determinant L is globally generated.

(3) Since F is globally generated we obtain a surjection $H^0(X,F) \otimes_k \mathcal{O}_X \longrightarrow F$. For a general $(R+1)$–dimensional linear subspace

$$W \subset H^0(X \times_k K, F \otimes_k K),$$

the corresponding homomorphism $W \otimes_K \mathcal{O}_{X\times_k K} \longrightarrow F \otimes_k K$ is surjective because X is smooth of dimension one.

(4) If $\det F \cong L$, then for any surjection $\pi : \mathcal{O}_X^{\oplus(R+1)} \longrightarrow F$, the kernel is L^{-1}. Thus, all those F (and a little bit more) are overparameterized by $\mathbb{P}(V)$ where $V := \mathrm{Hom}(L^{-1}, \mathcal{O}_X^{\oplus(R+1)})^\vee$. We consider the morphisms

$$X \xleftarrow{p} X \times \mathbb{P}(V) \xrightarrow{q} \mathbb{P}(V)$$

and have the universal short exact sequence on $X \times \mathbb{P}(V)$:

$$0 \longrightarrow L^{-1} \boxtimes \mathcal{O}(-1) \longrightarrow p^*\mathcal{O}_X^{\oplus(R+1)} \longrightarrow \mathcal{F} \longrightarrow 0\,.$$

Obviously, both $\mathbb{P}(V)$ and $\mathcal{F}$ are defined over k.

(5) We tensor the above short exact sequence of sheaves with p^*E, and apply the push forward q_* to $\mathbb{P}(V)$. Let

$$\longrightarrow H^1(X, E\otimes L^{-1}) \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \xrightarrow{\psi_E} H^1(X, E^{\oplus(R+1)}) \otimes \mathcal{O}_{\mathbb{P}(V)}$$
$$\longrightarrow R^1q_*(p^*E \otimes \mathcal{F}) \longrightarrow 0$$

be the resulting long exact sequence of sheaves on $\mathbb{P}(V)$. Since $E\otimes L^{-1}$ and $E^{\oplus(R+1)}$ are semistable vector bundles of negative degree, they have no global sections. Using the Riemann–Roch theorem, we get $h^1(X, E\otimes L^{-1}) = h^1(X, E^{\oplus(R+1)}) = rD+g-1-d$. The support Θ_E of $R^1q_*(p^*E\otimes F)$ is therefore the vanishing locus of the divisor $\det(\psi_E) \in H^0(\mathcal{O}_{\mathbb{P}(V)}(rD+g-1-d))$. Set theoretically Θ_E describes all short exact sequences

$$0 \longrightarrow L^{-1} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus(R+1)} \longrightarrow F_\alpha \longrightarrow 0$$

such that $h^1(E\otimes F_\alpha) > 0$. This includes all F_α which are not locally free, or not semistable. Popa's result [5, Theorem 5.3] implies that with our choices of the rank R the set $\Theta_E \subset \mathbb{P}(V)$ is a divisor, or equivalently, $\det(\psi_E) \neq 0$.

(6) Since k has infinitely many elements the k-rational points of the divisor Θ_E are a proper subset of $\mathbb{P}(V)(k)$. See also [3, p. 4, Proposition 1.3(a)]. □

4. Finite fields

In this section we consider a field k with q elements. We will need the additional number $M := \lceil \log_q(rD+g-1-d) \rceil$ along with the notation from Section 2.

Theorem 2. *Suppose k is a finite field with q elements, and let X be a smooth projective geometrically irreducible curve over k. For a vector bundle E of rank r and degree d over X, the following statements are equivalent:*

(i) *The vector bundle E is semistable.*
(ii) *There exists a vector bundle F on X defined over k such that $H^*(X, F \otimes E) = 0$.*
(iii) *For every field extension ℓ/k of degree at least M, there exists a vector bundle F' of rank R defined over ℓ such that $H^*(X \times_k \ell,\, F' \otimes (E \otimes_k \ell)) = 0$.*
(iv) *There exists a vector bundle F on X defined over k of rank $R \cdot M$ such that $H^*(X, F \otimes E) = 0$.*

Proof. We will show that (i) $\implies$ (iii) $\implies$ (iv). Also note that (iv) $\implies$ (ii) is obvious, and (ii) $\iff$ (i) was discussed in the introduction.

(i) $\implies$ (iii): We follow the proof of Theorem 1 in steps 1–5. To find a point outside the divisor $\Theta_E \subset \mathbb{P}(V)$, we pass to a field extension ℓ/k with at least $\deg(\Theta_E)$ elements. Thus, any field extension of degree at least M will do by our choice of M above. By [1, Lemma 2.2], there exists a point in $\mathbb{P}(V)(\ell)$ outside Θ_E. This point corresponds to a vector bundle F' defined over ℓ such that $H^*(X \times_k \ell,\, F' \otimes (E \otimes_k \ell)) = 0$.

(iii) $\implies$ (iv): We take a finite field extension ℓ/k of degree M. Now the field extension ℓ/k is Galois with Galois group $\mathrm{Gal} = \mathrm{Gal}(\ell/k)$. Setting $F := \bigoplus_{\gamma \in \mathrm{Gal}} \gamma^* F'$ we obtain a vector bundle of rank $R \cdot [\ell : k]$ which is defined over k, and $H^0(X,\, F \otimes E) = 0$. □

Remark: The rank of the cohomologically orthogonal bundle F in Theorem 1 (the case of infinite fields) is independent of the genus g of our curve X. However, the number M in Theorem 2 depends on g. Thus, in the case of a finite field the rank of F depends on g.

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